# CALCULUS ONE OVER COFFEE

## A Textbook and User-Friendly Guide to Calculus One

## Student Solutions Manual

## Odd Solutions

## CHAPTER 0

## CALCULUS MOTIVATION

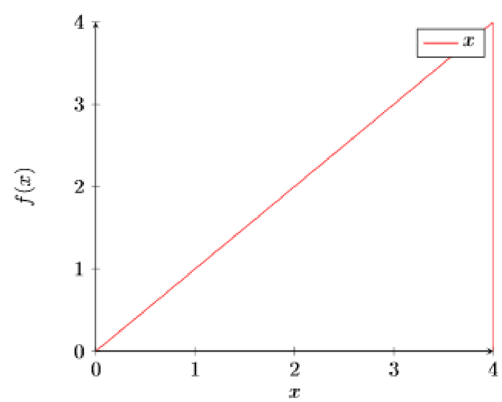
**EXERCISES:**

1) Find the average rates of change for the following functions: (i.e. in this case the average velocity):

   a) $s(t) = t^3 + 5t^2 - 1$, for $t = 1$ to $t = 5$: Average rate of change is $\frac{s(t_2)-s(t_1)}{t_2-t_1} = \frac{249}{4}$

   b) $s(t) = 2sint$ for $t = 0$ to $t = \frac{3\pi}{4}$: Average rate of change is $\frac{s(t_2)-s(t_1)}{t_2-t_1} = \frac{2\sin\frac{3\pi}{4}-2\sin 0}{\frac{3\pi}{4}-0} = \frac{4\sqrt{2}}{3\pi}$

3) Find the area of the following objects:



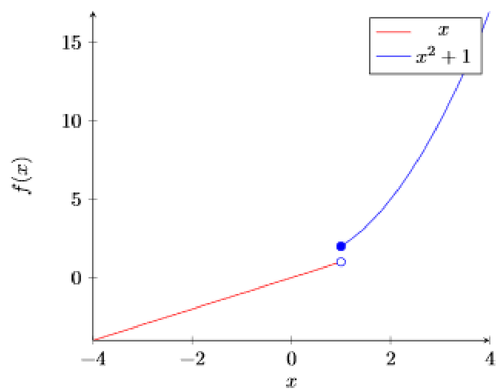

   a) $A = \frac{1}{2} \cdot 4 \cdot 4 = 4$
   b) $x^2 + y^2 = 4$. $A = \pi r^2 = \pi \cdot 2^2 = 4\pi$

**CHAPTER 1**
**SECTION 1**

**LIMITS**

**EXERCISES**

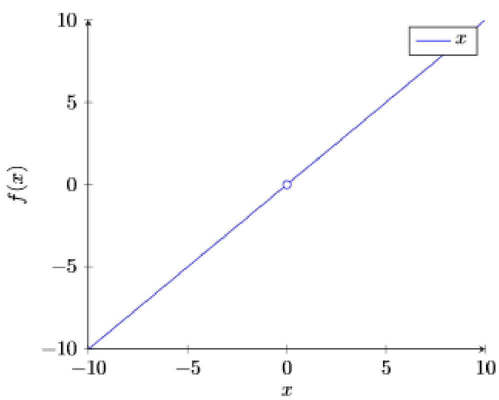
1)




Let us consider the graph above:

a) Find $\lim_{x \to 1^-} f(x)$: $= 0$
b) Find $\lim_{x \to 1^+} f(x)$: $= 2$
c) Find $\lim_{x \to 1} f(x)$: $= DNE$
d) Find $\lim_{x \to 0} f(x)$: $= 0$

3) Consider the graph:

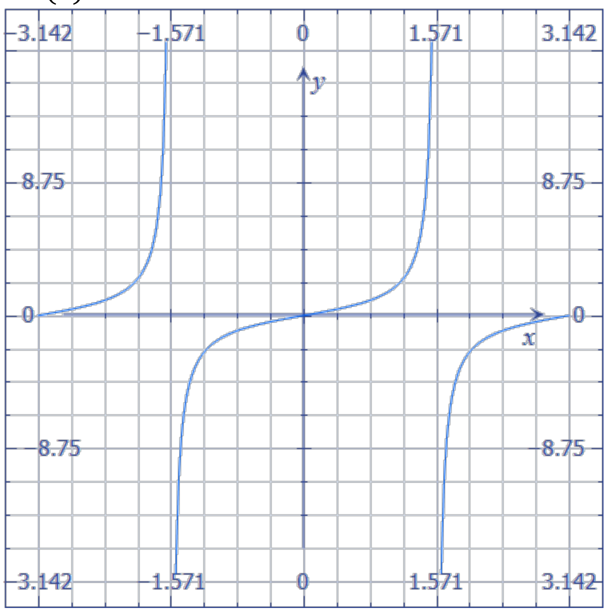



a) Find $\lim_{x \to 0^+} f(x)$: $= 0$

b) Find $\lim_{x \to 0^-} f(x): = 0$

c) Find $\lim_{x \to 0} f(x): = 0$

5) $\tan(x)$




a) Find $\lim_{x \to \pi^-} f(x): = 0$

b) Find $\lim_{x \to \pi^+} f(x): = 0$

c) Find $\lim_{x \to \pi} f(x): = 0$

Find the following infinite limits:

7) $\lim_{x \to 1^+} \frac{3}{x^2-1}: = \infty$

9) $\lim_{x \to 1^+} \frac{x+1}{x-1} := \infty$

11) $\lim_{x \to \frac{\pi}{2}^+} \tan(x): = -\infty$

Find the following vertical asymptotes:

13) $f(x) = \frac{1}{x-7}: \ x = 7$

15) $f(x) = \frac{x-2}{x^2+2x-8}: \ x^2 + 2x - 8 = 0 \rightarrow \quad (x+4)(x-2) = 0, \ \ x = 2 \text{ is a hole. } x = -4 \text{ is our vertical asymptote.}$

17) $f(x) = tanx, 0 \leq x \leq 2\pi \rightarrow \quad \cos x = 0 \rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2}$

**CHAPTER 1**
**SECTION 2**

**LIMITS USING LIMIT LAWS**
**(ALGEBRAIC LIMITS)**

**Exercises:**

Find the following limits (if they exist):

1) $\lim_{x\to 1} x^2 + 2x - 5: \ = -2$

3) $\lim_{x\to -1} \frac{x^2-1}{x+1}: \ = \lim_{x\to -1} \frac{(x+1)(x-1)}{x+1} = \lim_{x\to -1} x - 1 = -2$

5) $\lim_{x\to -2} \frac{2x^2+3x-2}{x+2}: \ = \lim_{x\to -2} \frac{(2x-1)(x+2)}{x+2} = \lim_{x\to -2} 2x - 1 = -5$

7) $\lim_{x\to 1} \frac{x^2-1}{x^4-1}: \ = \lim_{x\to 1} \frac{x^2-1}{(x^2-1)(x^2+1)} = \lim_{x\to 1} \frac{1}{x^2+1} = \frac{1}{2}$

9) $\lim_{x\to 1} \frac{x^3-2x^2+x}{x-1}: \ = \lim_{x\to 1} \frac{x(x^2-2x+1)}{x-1} = \lim_{x\to 1} \frac{x(x-1)^2}{x-1} = \lim_{x\to 1} x(x-1) = 0$

11) $\lim_{x\to 0^+} \frac{\sqrt{x+5}-\sqrt{5}}{x}: \ = \lim_{x\to 0^+} \frac{\sqrt{x+5}-\sqrt{5}}{x} \cdot \frac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}} = \lim_{x\to 0^+} \frac{x}{x(\sqrt{x+5}-\sqrt{5})} = \lim_{x\to 0^+} \frac{1}{\sqrt{x+5}-\sqrt{5}} = \infty$

13) $\lim_{t\to 0^-} \frac{3-\sqrt{9-t}}{t}: \ = \lim_{t\to 0^-} \frac{3-\sqrt{9-t}}{t} \cdot \frac{3+\sqrt{9-t}}{3+\sqrt{9-t}} = \lim_{t\to 0^-} \frac{t}{t(3-\sqrt{9-t})} = \lim_{t\to 0^-} \frac{1}{(3-\sqrt{9-t})} = -\infty$

15) $\lim_{t\to 0} \frac{(x+t)^2-x^2}{t}: \ = \lim_{t\to 0} \frac{x^2+2xt+t^2-x^2}{t} = \lim_{t\to 0} \frac{t(2x+1)}{t} = 2x + 1$

17) Use the Squeeze Theorem to find the following limits:

a) Let $x^2 + 1 \le f(x) \le e^x$ for $x \ge 0$ Find $\lim_{x \to 0} f(x)$: $\lim_{x \to 0} x^2 + 1 = 1$. $\lim_{x \to 0} e^x = 1$.
Therefore, $\lim_{x \to 0} f(x) = 0$ by the Squeeze Theorem.

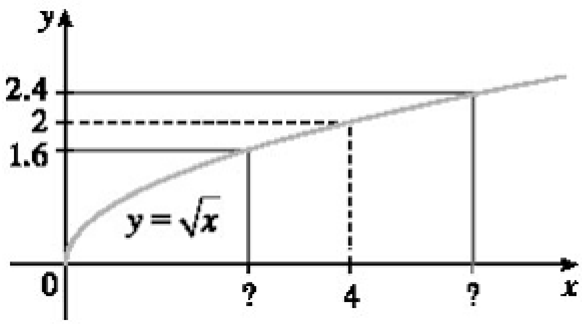
b) Find $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$: First we find bounds: We know: $-1 \le \cos\left(\frac{1}{x}\right) \le 1$.

Therefore, $-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2$. Next, we take limits: $\lim_{x \to 0} -x^2 = 0$, and $\lim_{x \to 0} x^2 = 0$.
Therefore, $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$ by the Squeeze Theorem.

# CHAPTER 1
# SECTION 3

## PRECISE DEFINITION OF A LIMIT

**EXERCISES:**

1) 


Find the values for $\delta$ given the values of $\epsilon$. $\delta = 1.6^2, 2.4^2 = 2.56, 5.76$. $\delta$ must be the smaller of the two values to get the proper value for $\epsilon$, therefore $\delta$ must equal 2.56.

Find the values of $\delta$, for the following function if $\epsilon = .1$ on both sides:

3) $\lim_{x \to 2}(\frac{1}{2}x + 1) = 2$: We first start with $0 < |x - a| < \delta$. This is the "if", which means it is the given. We always assume this part is true before proceeding further. In our case, we have $0 < |x - 2| < \delta$. We will next look at the fact that we want $|f(x) - L| < .1$. Let us manipulate $|f(x) - L|$. We have $f(x) = \frac{1}{2}x + 1$. We have $L = 2$. So $|f(x) - L| < .1 \to \left|\left(\frac{1}{2}x + 1\right) - 2\right| < .1$. This implies $\left|\frac{1}{2}x - 1\right| < .1 \to \frac{1}{2}|x - 2| < .1$. Since $0 < |x - 2| < \delta$, then it is easy to see that $\delta$ must equal .2

Prove the following limits using our Precise Definition of a limit:

5) $\lim_{x \to 2}(6x - 6) = 6$: **PRELIMINARY WORK:** We start by manipulating $|(6x - 6) - 6| = |6x - 12| = 6|x - 2|$. Recall that $0 < |x - 2| < \delta$ is the if (or the given). This is the part we have to assume is true. So if $0 < |x - 2| < \delta$ is true, then $0 < 6|x - 2| < 6\delta$. We want $6|x - 2| < \epsilon$. We choose $\delta = \frac{\epsilon}{6}$. We have $0 < 6|x - 2| < 6\delta \to 0 < 6|x - 2| < 6\left(\frac{\epsilon}{6}\right) \to 0 < 6|x - 2| < \epsilon$.

**FORMAL PROOF:** For every $\epsilon > 0,$ there exists a $\delta > 0$ such that If $0 < |x-2| < \delta$, then $0 < 6|x-2| < 6\delta \rightarrow |(6x-6)-6| < 6\delta$. But we let $\delta = \frac{\epsilon}{6}$. Therefore-$|(6x-6)-6| < \frac{6\epsilon}{6} \rightarrow |(6x-6)-6| < \epsilon$. Q.E.D.

7) $\lim_{x \to 3}(5\,x+1) = 16$: **PRELIMINARY WORK:** We start by manipulating $|(5x+1)-16| = |5x-15| = 5|x-3|$. Recall that $0 < |x-3| < \delta$ is the if (or the given). This is the part we have to assume is true. So if $0 < |x-3| < \delta$ is true, then $0 < 5|x-3| < 5\delta$. We want $5|x-3| < \epsilon$. We choose $\delta = \frac{\epsilon}{5}$. We have $0 < 5|x-3| < 5\delta \rightarrow 0 < 5|x-3| < 5\left(\frac{\epsilon}{5}\right) \rightarrow$ $0 < 5|x-3| < \epsilon$.

**FORMAL PROOF:** For every $\epsilon > 0,$ there exists a $\delta > 0$ such that If $0 < |x-3| < \delta$, then

$0 < 5|x-3| < 5\delta \rightarrow |(5x+1)-16| < 5\delta$. But we let $\delta = \frac{\epsilon}{5}$. Therefore-$|(5x+1)-16| < \frac{5\epsilon}{5} \rightarrow |(3x-2)-4| < \epsilon$. Q.E.D.

9) $\lim_{x \to \frac{1}{5}}(5\,x+2) = 3$: **PRELIMINARY WORK:** We start by manipulating $|(5x+2)-3| = |5x-1| = 5\left|x-\frac{1}{5}\right|$. Recall that $0 < \left|x-\frac{1}{5}\right| < \delta$ is the if (or the given). This is the part we have to assume is true. So if $0 < \left|x-\frac{1}{5}\right| < \delta$ is true, then $0 < 5\left|x-\frac{1}{5}\right| < 5\delta$. We want $5\left|x-\frac{1}{5}\right| < \epsilon$. We choose $\delta = \frac{\epsilon}{5}$. We have $0 < 5\left|x-\frac{1}{5}\right| < 5\delta \rightarrow 0 < 5\left|x-\frac{1}{5}\right| < 5\left(\frac{\epsilon}{5}\right) \rightarrow$ $0 < 5\left|x-\frac{1}{5}\right| < \epsilon$.

**FORMAL PROOF:** For every $\epsilon > 0,$ there exists a $\delta > 0$ such that If $0 < \left|x-\frac{1}{5}\right| < \delta$, then $0 < 5\left|x-\frac{1}{5}\right| < 5\delta \rightarrow |(5x+2)-3| < 5\delta$. But we let $\delta = \frac{\epsilon}{5}$. Therefore-$|(5x+2)-3| < \frac{5\epsilon}{5} \rightarrow |(5x+2)-3| < \epsilon$. Q.E.D.

11) $\lim_{x \to 2} \mathrm{c} = c$: $|c-c| = 0$. This works for any $\delta$, since $\delta > 0$, and $\epsilon > 0$. We found $|f(x)-L| = 0$ which is always less than $\epsilon$, since $\epsilon > 0$. You may choose any $\delta > 0$.

13) $\lim_{x \to 1} 2\mathrm{x}^2 = 2$: We have $|2x^2-2| = 2|x+1||x-1|$. We know $0 < |x-1| < \delta$. Then $2|x+1||x-1| < 2|x+1|\delta$. We need to find a positive constant $C$, such that $|x+1| < \frac{C}{2}$. Then

$|x+1||x-1| < \frac{C\delta}{2}$. Let $\delta = \frac{2\epsilon}{C}$. We need to find $C$. Recall, since we are proving a limit, it means that we are close to $x = 1$. Therefore, let us assume we are within one value away from $x = 1$. Therefore $|x-1| < 1$. So $0 < x < 2 \rightarrow \ 1 < x+1 < 3$. And, $|x+1| < 3$. So we can now choose $C$ to be 3. So $|x-1| < \frac{2\epsilon}{3}$. But we also had the previous inequality $|x-1| < 1$. Recall that $\delta$ must be the smallest value to ensure that we will be within an $\epsilon$ distance away from our limit $L$. Therefore $\delta$ must be the minimum of $\frac{2\epsilon}{3}$ and 1.

**FORMAL PROOF:** For every $\epsilon > 0$, there exists a $\delta > 0$ such that If $0 < |x-1| < \delta$, then $|x-1| < 1 \rightarrow 0 < x < 2 \rightarrow \ 1 < x+1 < 3 \rightarrow \ |x+1| < 3$. We also have $2|x-1| < \frac{2\epsilon}{3}$ so

$|2x^2 - 2|$ = $2|x+1||x-1| < 2C\delta \rightarrow 2|x+1||x-1| < \frac{3}{2}\delta$. We let $\delta = \min\left\{1, \frac{2\epsilon}{3}\right\} \rightarrow$

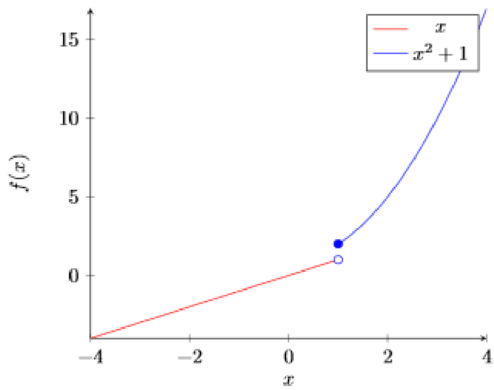
$2|x+1||x-1| < \frac{3}{2} \cdot \frac{2\epsilon}{3} = \epsilon$, Q.E.D.

**CHAPTER 1**
**SECTION 4**

**CONTINUITY**

**EXERCISES:**

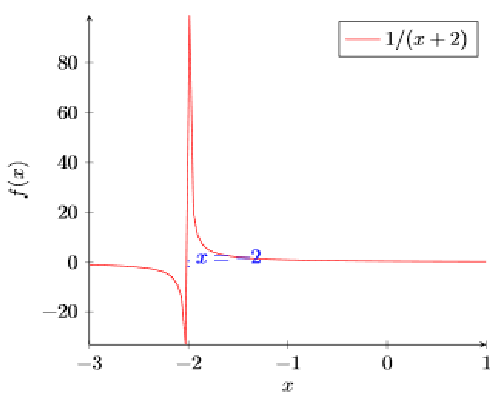
1) Refer to the graph below (from Exercise 1) of Section 1.1):




a) Is $f$ continuous at $x = 1$? Why or why not? No, $\lim_{x\to 1} f(x)\ DNE$

b) Is $f$ continuous at $x = 3$? Why or why not? Yes, $\lim_{x\to 3} f(x) = f(3) = 10$

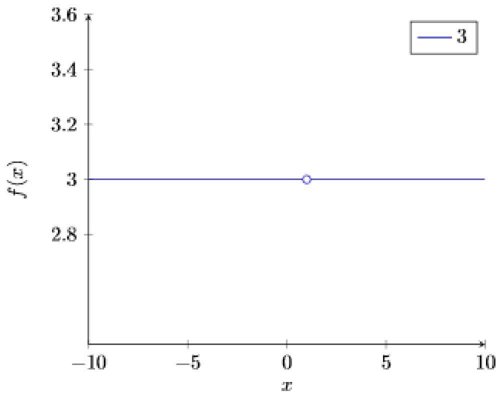
3) Refer to the graph below (from Exercise 3) of Section 1.1):




a) Is $f$ continuous at $x = -2$? Why or why not? No, $f(-2) = DNE$. Also, $\lim_{x\to 2} f(x) = DNE$

b) Is $f$ continuous at $x = 0$? Why or why not? Yes, $\lim_{x\to 0} f(x) = f(0) = \frac{1}{2}$.

5)

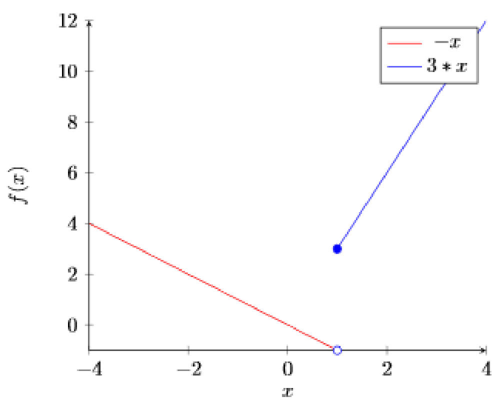



a) Is $f$ continuous at $x = 0$? Why or why not? Yes, $\lim_{x \to 0} f(x) = f(0) = 3$.

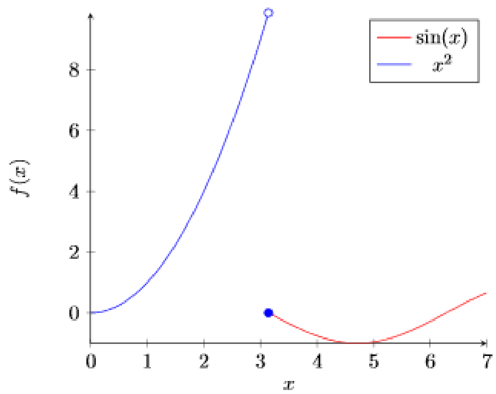
b) Is there a removable discontinuity? If so, where? Yes, at $\approx x = 1$

Explain why $f$ is discontinuous at the given value for $x = a$. Sketch the graph.

7) $f(x) = \begin{cases} -x, & x < 1 \\ 3x, & x \geq 1 \end{cases}$ $a = 1$. $f$ is discontinuous at $a = 1$, because $\lim_{x \to 1} f(x) = DNE$.




9) $f(x) = \begin{cases} x^2, x < \pi \\ \sin x, x \geq \pi \end{cases}$ $a = \pi$. $f$ is discontinuous at $a = \pi$, because $\lim_{x \to \pi} f(x) = DNE$.




Use the Intermediate Value Theorem to show a root exists for the equation in the given interval:

11) $x^3 - x + 1 = 0,\ (-2,0)$: Let $f(x) =:\ x^3 - x + 1$. $f$ is a polynomial, which is continuous everywhere. Therefore $f$ is continuous on $[-2,0]$. Also $f(-2) \neq f(0)$. Therefore, the Intermediate Value Theorem applies. This means there exists a value $x = c$ such that $c$ is in $(-2,0)$ such that $f(c) = N$. Since $f(-2) = -5$, which is negative, and $f(0) = 1$ which is positive, we conclude there is a value between -2 and 0 such that $f(c) = 0$. Therefore, there is at least one root in the interval (-2,0).

13) $x^5 - 2x^2 = 1,$ (0,2): Let $f(x) =:\ x^5 - 2x^2 - 1$. $f$ is a polynomial, which is continuous everywhere. Therefore $f$ is continuous on $[0,2]$. Also $f(0) \neq f(2)$. Therefore, the Intermediate Value Theorem applies. This means there exists a value $x = c$ such that $c$ is in $(0,2)$ such that $f(c) = N$. Since $f(0) = -1$, which is negative, and $f(2) = 23$ which is positive, we conclude there is a value between 0 and 2 such that $f(c) = 0$. Therefore, there is at least one root in the interval (0,2).

**CHAPTER 1**
**SECTION 5**

**LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES**

**EXERCISES:**

Find the following limits (if they exist), or show they do not exist:

1) $\lim_{x\to\infty} \frac{1}{x+6}:\ = 0$

3) $\lim_{x\to\infty} \frac{3x}{x^2+x}$: Is a form of $\frac{\infty}{\infty}$ which is indeterminate. Divide all terms by $x^2$ to get $\lim_{x\to\infty} \frac{\frac{3}{x}}{1+\frac{1}{x}} = \frac{0}{1} = 0.$

5) $\lim_{x\to\infty} \frac{x^2+5x-27}{4x^2+9}$: Is a form of $\frac{\infty}{\infty}$ which is indeterminate. Divide all terms by $x^2$ to get $\lim_{x\to\infty} \frac{1+\frac{5}{x}-\frac{27}{x^2}}{4+\frac{9}{x^2}} = \frac{1}{4}.$

7) $\lim_{x\to\infty} \frac{x^{\frac{1}{2}}-7}{x^{\frac{1}{4}}+8}$: Is a form of $\frac{\infty}{\infty}$ which is indeterminate. Divide all terms by $x^{\frac{1}{4}}$ to get $\lim_{x\to\infty} \frac{x^{\frac{1}{4}}-\frac{7}{x^{\frac{1}{4}}}}{1+\frac{8}{x^{\frac{1}{4}}}} = \infty.$

9) $\lim_{x\to-\infty} \frac{4x^4}{2x^4+6x^3-7x^2+11}$: Is a form of $\frac{\infty}{\infty}$ which is indeterminate. Divide all terms by $x^4$ to get $\lim_{x\to\infty} \frac{4}{2+\frac{6}{x}-\frac{7}{x^2}+\frac{11}{x^4}} = \frac{4}{2} = 2.$

11) $\lim_{x\to\infty} \frac{\sqrt{x^4+3x^2}}{9x^2+7x-5}$: Is a form of $\frac{\infty}{\infty}$ which is indeterminate. Divide all terms by $x^2$ to get $\lim_{x\to\infty} \frac{\sqrt{\frac{x^4}{x^4}+\frac{3x^2}{x^4}}}{9+\frac{7}{x}-\frac{5}{x^2}} = \frac{1}{9}.$

13) $\lim_{x\to\infty} \frac{\cos x}{x-4}$: (Hint: Use the Squeeze Theorem). First: $-1 \le \cos x \le 1$. Next we divide all terms by $x-4$ to get: $-\frac{1}{x-4} \le \frac{\cos x}{x-4} \le \frac{1}{x-4}$. We take the limit of the LHS and the RHS: $\lim_{x\to\infty} -\frac{1}{x-4} = 0$, And $\lim_{x\to\infty} \frac{1}{x-4} = 0$. Therefore, by the Squeeze Theorem $\lim_{x\to\infty} \frac{\cos x}{x-4} = 0.$

15) $\lim_{x\to-\infty} x-3:\ = -\infty$

17) $\lim_{x\to\infty} \ln x:\ = \infty$

Find both the vertical and horizontal asymptotes for the following functions:

19) $f(x) = \frac{x-1}{x^2+5x-6}$: To find the vertical asymptote(s): We set $x^2 + 5x - 6 = 0 \rightarrow$ $(x+6)(x-1) = 0 \rightarrow \quad x = -6$ is a vertical asymptote, and $x = 1$ is a hole.

To find the horizontal asymptote: $\lim_{x \to \infty} \frac{x-1}{x^2+5x-6} = \frac{\frac{1}{x} - \frac{1}{x^2}}{1 + \frac{5}{x} - \frac{6}{x^2}} = 0$. So $y = 0$ is the horizontal asymptote.

21) $f(x) = \frac{x^3+x^2-3x}{13x+9}$: To find the vertical asymptote(s): We set $13x + 9 = 0 \rightarrow \quad x = -\frac{9}{13}$

There is no horizontal asymptote as $\lim_{x \to \infty} \frac{x^3+x^2-3x}{13x+9} = \infty$

**CHAPTER 1**
**SECTION 6**

**DERIVATIVES**

**EXERCISES:**

Find $f(a)$ for $f(x)$ for the given value of $a$, by using i) $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$ and ii) $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$

1) $f(x) = -x^2,\ x = 3$: i) $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{x \to 3} \frac{f(x)-f(3)}{x-3} = \lim\limits_{x \to 3} \frac{-x^2-(-3^2)}{x-3} = \lim\limits_{x \to 3} \frac{9-x^2}{x-3} = -\lim\limits_{x \to 3} \frac{(x+3)(x-3)}{x-3} = -6$. Ii) $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim\limits_{h \to 0} \frac{-(3+h)^2-(-3^2)}{h} = \lim\limits_{h \to 0} \frac{-(3^2+2\cdot 3\cdot h+h^2)+9}{h} = \lim\limits_{h \to 0} \frac{h(-6-h)}{h} = -6$

3) $f(x) = -x^3,\ x = 2$: i) $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{x \to 2} \frac{-x^3-(-2^3)}{x-2} = \lim\limits_{x \to 2} \frac{-(x-2)(x^2+2x+4)}{x-2} = -12$.
Ii) $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim\limits_{h \to 0} \frac{-(2+h)^3-(-2^3)}{h} = \lim\limits_{h \to 0} \frac{-(2^8+3\cdot 2^2\cdot h+3\cdot 2\cdot h^2+h^3)+8}{h} = -\lim\limits_{h \to 0} \frac{h(12+6h+h^2)}{h} = -12$

5) Find the equation of the tangent line for the problems in number 1): $m = -6$. Also, $y = -9$. Therefore, the equation of the tangent line is: $y + 9 = -6(x-3) \to \quad y = -6x + 9$.

Find the derivatives for the following functions: (i.e. find $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$.)

7) $f(x) = x^2 - 3$: $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h \to 0} \frac{(x+h)^2-3-(x^2-3)}{h} = \lim\limits_{h \to 0} \frac{x^2+2xh+h^2-3-x^2+3}{h} = \lim\limits_{h \to 0} \frac{h(2x+h)}{h} = 2x$.

9) $f(x) = \frac{1}{x-1}$: $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h \to 0} \frac{\frac{1}{(x+h)-1} - \frac{1}{x-1}}{h} = \lim\limits_{h \to 0} \frac{\frac{x-1}{(x+h-1)(x-1)} - \frac{x+h-1}{(x+h-1)(x-1)}}{h} = \lim\limits_{h \to 0} \frac{\frac{-h}{(x+h-1)(x-1)}}{h} = \lim\limits_{h \to 0} \frac{-h}{(x+h-1)(x-1)} \cdot \frac{1}{h} = \lim\limits_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = -\frac{1}{(x-1)^2}$

11) $f(x) = x^2 + 2$: $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h \to 0} \frac{(x+h)^2+2-(x^2+2)}{h} = \lim\limits_{h \to 0} \frac{x^2+2xh+h^2+2-x^2-2}{h} = \lim\limits_{h \to 0} \frac{h(2x+h)}{h} = 2x$.

13) $f(x) = 5x^2 + 3x - 7$: $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x) = \lim\limits_{h \to 0} \frac{5(x+h)^2+3(x+h)-7-(5x^2+3x-7)}{h} = \lim\limits_{h \to 0} \frac{5x^2+10xh+5h^2+3x+3h-7-5x^2-3x+7}{h} = \lim\limits_{h \to 0} \frac{h(10x+5h+3)}{h} = 10x + 3$.

15) $f(x)=\frac{2}{x+3}$: $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\frac{\frac{2}{(x+h)+3}-\frac{2}{x+3}}{h}=\lim\limits_{h\to 0}\frac{\frac{2(x+3)}{(x+h+3)(x+3)}-\frac{2(x+h+3)}{(x+h+3)(x+3)}}{h}=$

$\lim\limits_{h\to 0}\frac{\frac{-2h}{(x+h+3)(x+3)}}{h}=\lim\limits_{h\to 0}\frac{-2h}{(x+h+3)(x+3)}\cdot\frac{1}{h}=\lim\limits_{h\to 0}\frac{-2}{(x+h+3)(x+3)}=-\frac{2}{(x+3)^2}$

17) $f(x)=\sqrt{x+3}$: $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\frac{\left(\sqrt{x+h+3}-\sqrt{x+3}\right)}{h}=\lim\limits_{h\to 0}\frac{\left(\sqrt{x+h+3}-\sqrt{x+3}\right)}{h}\cdot\frac{\left(\sqrt{x+h+3}+\sqrt{x+3}\right)}{\left(\sqrt{x+h+3}+\sqrt{x+3}\right)}=$

$\lim\limits_{h\to 0}\frac{x+h+3-(x+3)}{h\left(\sqrt{x+h+3}+\sqrt{x+3}\right)}=\lim\limits_{h\to 0}\frac{h}{h\left(\sqrt{x+h+3}+\sqrt{x+3}\right)}=\frac{1}{2\sqrt{x+3}}.$

19) $f(x)=\frac{1}{\sqrt{x-1}}$: : $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\frac{\frac{1}{\sqrt{x+h-1}}-\frac{1}{\sqrt{x-1}}}{h}=\lim\limits_{h\to 0}\frac{\frac{\sqrt{x-1}}{\sqrt{x-1}\sqrt{x+h-1}}-\frac{\sqrt{x+h-1}}{\sqrt{x-1}\sqrt{x+h-1}}}{h}=$

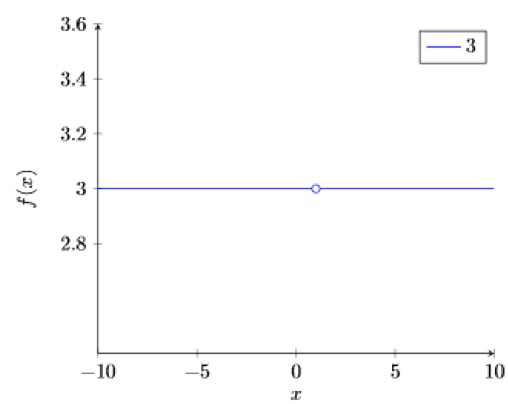
$\lim\limits_{h\to 0}\frac{\sqrt{x-1}-\sqrt{x+h-1}}{\sqrt{x-1}\sqrt{x+h-1}}\cdot\frac{1}{h}=\lim\limits_{h\to 0}\frac{1}{h}\cdot\frac{\sqrt{x-1}-\sqrt{x+h-1}}{\sqrt{x-1}\sqrt{x+h-1}}\cdot\frac{\sqrt{x-1}+\sqrt{x+h-1}}{\sqrt{x-1}+\sqrt{x+h-1}}=$

$\lim\limits_{h\to 0}\frac{1}{h}\cdot\frac{x-1-(x+h-1)}{\left(\sqrt{x-1}\sqrt{x+h-1}\right)\cdot\left(\sqrt{x-1}+\sqrt{x+h-1}\right)}$

$=\lim\limits_{h\to 0}\frac{1}{h}\cdot\frac{-h}{\left(\sqrt{x-1}\sqrt{x+h-1}\right)\cdot\left(\sqrt{x-1}+\sqrt{x+h-1}\right)}=\frac{-1}{(x-1)\cdot 2\sqrt{x-1}}$

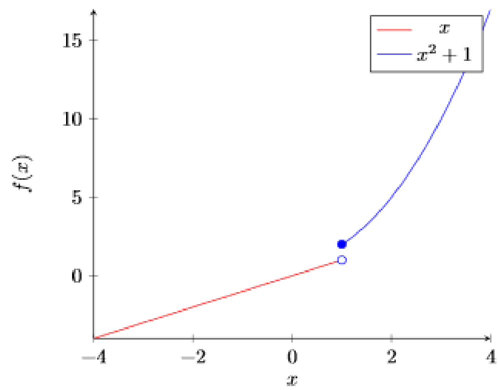
$=-\frac{1}{2(x-1)^{\frac{3}{2}}}$

21) State where (meaning at which x-values) the following graphs are not differentiable, and why:

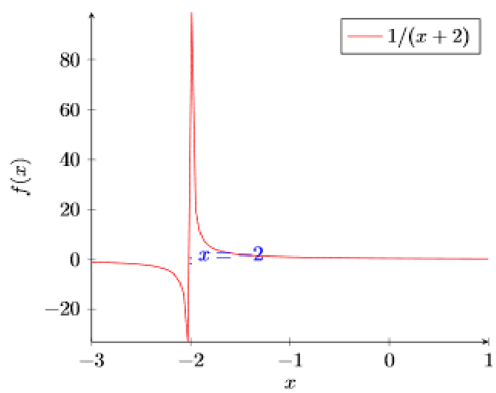



a)

Not differentiable at $x=1$, because it is not continuous.

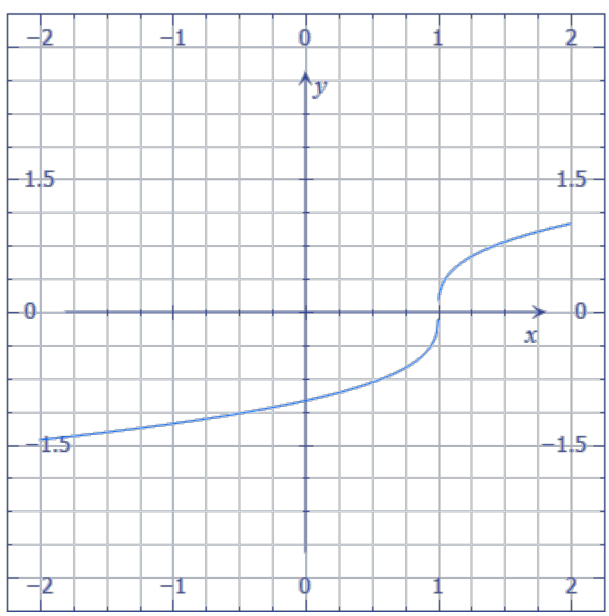



b)
Not differentiable at $x = 1$, because it is not continuous.

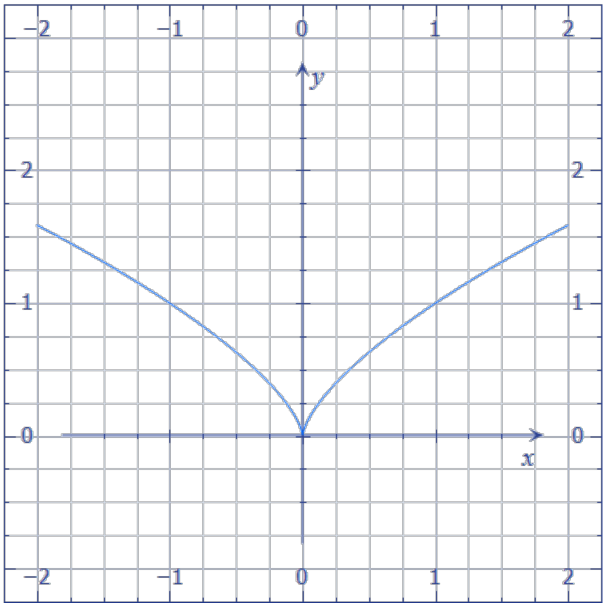



c)
Not differentiable at $x = -2$, because it is not continuous.

d)




Not continuous at $x = 1$, because it has a vertical tangent.

e) 


Not continuous at $x = 0$, because there is a sharp corner.

**CHAPTER 2**
**SECTION 1**

**DIFFERENTIATION**
**POWER RULE AND SUM/DIFFERENCE RULES:**

**EXERCISES:**

Find the following derivatives by using The Power Rule, The Constant Multiple Rule, The Sum/Difference Rule, or any combination thereof:

1) $f(x) = x^5$: $f'(x) = 5x^4$

3) $y = \frac{1}{x^3} = x^{-3}$: $f'(x) = -3x^{-4} = -\frac{3}{x^4}$

5) $f(x) = x^{-\frac{5}{6}}$: $f'(x) = -\frac{5}{6}x^{-\left(\frac{11}{6}\right)} = -\frac{5}{6x^{\frac{11}{6}}}$

7) $f(x) = \frac{1}{2}x^{-\frac{1}{2}}$: $f'(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x^{\frac{3}{2}}}$

9) $f(x) = x^6 + 4x^5 - 7x^3 + 6x - 12$: $f'(x) = 6x^5 + 20x^4 - 21x^2 + 6$

11) $f(x) = 4x^{\frac{1}{2}} - 12x^5 - \frac{1}{x^4} + \frac{3}{x^2} - 2x^3$: $f'(x) = 2x^{-\frac{1}{2}} - 60x^4 + 4x^{-5} - 6x^{-3}$

13) $f(x) = (x^2 - 1)(x + 4) = x^3 + 4x^2 - x - 4$: $f'(x) = 3x^2 + 8x - 1$

15) $f(x) = x^4(x^2 + 2x - 7) = x^6 + 2x^5 - 7x^4$: $f'(x) = 6x^5 + 10x^4 - 28x^3$

17) $f(x) = \frac{x^{\frac{1}{2}} - 2x^2 + 4x + 6}{x^2} = x^{-\frac{3}{2}} - 2 + 4x^{-1} + 6x^{-2}$: $f'(x) = -\frac{3}{2}x^{-\frac{5}{2}} - 4x^{-2} - 12x^{-3}$

Find the equation of the tangent line to the curve at the given point (or value).

19) $f(x) = x^2 + 9,\ (1{,}10)$: $f'(x) = 2x \rightarrow \quad f'(1) = 2.$ Then $y - 10 = 2(x - 1) \rightarrow \quad y = 2x + 8$

21) $f(x) = \sqrt{x},\ x = 4$: $f'(x) = \frac{1}{2\sqrt{x}} \rightarrow \quad f'(4) = \frac{1}{4}$. Then $y - 2 = \frac{1}{4}(x - 4) \rightarrow \quad y = \frac{1}{4}x + 1$

Find the following higher order derivatives:

23) Find $f^{(5)}(x)$ for $f(x) = x^6 - 2x^5 + 3x^4 + 6x - 9$: $f'(x) = 6x^5 - 10x^4 + 12x^3 + 6,\ \ f''(x) = 30x^4 - 40x^3 + 36x^2,\ \ f'''(x) = 120x^3 - 120x^2 + 72x,\ \ f^4(x) = 360x^2 - 240x + 72,\ \ f^5(x) = 720x - 240.$

25) Find $\frac{d^2y}{dx^2}$ for $y = x^{\frac{1}{2}} - \frac{3}{x} + x$: $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + 3x^{-2} + 1,\ \ \frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\left(\frac{3}{2}\right)} - 6x^{-3}$

27) Let $s(t) = 2t^2 + 4t - 2$ by the distance function where $s$ is in meters, and $t$ is in seconds.

$s'(t) = v(t) = 4t + 5, \quad s''(t) = v'(t) = a(t) = 4$

a) Find the velocity and acceleration when $t = 1$ second, $v(1) = 9,\ \ a(1) = 4$

b) Find the velocity and acceleration when $t = 3$ seconds, $v(3) = 17,\ \ a(3) = 4$

**CHAPTER 2**
**SECTION 2**

**DIFFERENTIATION**
**DERIVATIVES OF THE NATURAL EXPONENTIAL FUNCTION AND THE TRIGONOMETRIC FUNCTIONS OF SINE AND COSINE**

**EXERCISES:**

Find the following derivatives:

1) $f(x) = 12e^x - 4\sin x + \frac{1}{x^{\frac{4}{3}}}$: $f'(x) = 12e^x - 4\cos x - \frac{4}{3}x^{-\frac{7}{3}}$

3) $f(x) = 3\sin x + 14e^x$: $f'(x) = 3\cos x + 14e^x$

Find the equation of the tangent line to the curve at the given x-value:

5) $f(x) = \cos x - \sin x, x = 2\pi$: $f'(x) = -\sin x + \cos x$, $f'(2\pi) = 1$. $f(2\pi) = 1$. $y - 1 = 1(x - 2\pi) \rightarrow \quad y = x - 2\pi + 1$

Find the following limits:

7) $\lim_{x \to 0} \frac{3\sin x}{8x} = \frac{3}{8}\lim_{x \to 0} \frac{\sin x}{x} = \frac{3}{8} \cdot 1 = \frac{3}{8}$

**CHAPTER 2**
**SECTION 3**

**DIFFERENTIATION**
**PRODUCT AND QUOTIENT RULES**
**OTHER TRIGONOMETRIC DERIVATIVES**

**EXERCISES:**

Differentiate:

1) $f(x) = x^2(2x^3 + 4x^2 + 5x - 6)$: $f'(x) = x^2(6x^2 + 8x + 5) + 2x(2x^3 + 4x^2 + 5x - 6)$

3) $f(x) = (e^x + 10x^2 - 4x)(\sin x - 5x^2)$: $f'(x) = (e^x + 10x^2 - 4x)(\cos x - 10x) + (e^x + 20x - 4)(\sin x - 5x^2)$

5) $y = \left(\csc x + \frac{1}{2}e^x + \frac{1}{x^{\frac{1}{7}}}\right)(2\sin x + \cot x + 3x^4)$: $\frac{dy}{dx} = (\csc x + \frac{1}{2}e^x + \frac{1}{x^{\frac{1}{7}}})(2\cos x - \csc^2 x + 12x^3) + (-\csc x \cot x + \frac{1}{2}e^x - \frac{1}{7}x^{-\frac{8}{7}})\,(2\sin x + \cot x + 3x^4)$

7) $f(x) = \frac{x^2+2}{2x-4}$: $f'(x) = \frac{(2x-4)(2x)-2(x^2+2)}{(2x-4)^2} = \frac{4x^2-8x-2x^2-4}{(2x-4)^2} = \frac{2x^2-8x-4}{(2x-4)^2} = \frac{x^2-4x-2}{4(x-2)^2}$

9) $y = \frac{e^x-2x^2+x^{-3}}{4x^2+9x}$: $\frac{dy}{dx} = \frac{(4x^2+9x)(e^x-4x-3x^{-4})-(8x+9)(e^x-2x^2+x^{-3})}{(4x^2+9x)^2}$

11) $f(x) = \frac{\tan x+\cos x}{e^x-\csc x}$: $f'(x) = \frac{(e^x-\csc x)(\sec^2 x-\sin x)-(e^x+\csc x\cot x)(\tan x+\cos x)}{(e^x-\csc x)^2}$

13) $y = \frac{e^x-\sin x}{e^x\cos x+4x^4}$: $\frac{dy}{dx} = \frac{(e^x\cos x+4x^4)(e^x-\cos x)-(-e^x\sin x+e^x\cos x+16x^3)(e^x-\sin x)}{(e^x\cos x+4x^4)^2}$

15) $y = \frac{1}{t^2+\sin t+t^{\frac{2}{3}}}$: $\frac{dy}{dx} = \frac{(t^2+\sin t+t^{\frac{2}{3}})\cdot 0-(2t+\cos t+\frac{2}{3}t^{-\frac{1}{3}})\cdot 1}{\left(t^2+\sin t+t^{\frac{2}{3}}\right)^2} = \frac{(2t+\cos t+\frac{2}{3}t^{-\frac{1}{3}})}{\left(t^2+\sin t+t^{\frac{2}{3}}\right)^2}$

17) $f(x) = \frac{cx}{c\sin x-cx^4+cx}$ where $c$ is a constant. $f'(x) = \frac{(c\sin x-cx^4+cx)\,c-(c\cos x-4cx^3+c)\,cx}{(c\sin x-cx^4+cx)^2}$

19) $f(x) = \frac{12x^3}{e^x+20x}$: $f'(x) = \frac{(e^x+20x)\cdot 36x^2-(e^x+20)\cdot 12x^3}{(e^x+20x)^2}$

Find $\frac{d^2y}{dx^2}$ for the following functions:

21) $y = \frac{x^2+4}{4x-9}$: $\frac{dy}{dx} = \frac{(4x-9)(2x)-4(x^2+4)}{(4x-9)^2} = \frac{8x^2-18x-4x^2-16}{(4x-9)^2} = \frac{4x^2-18x-16}{(4x-9)^2}$, $\frac{d^2y}{dx^2} =$
$\frac{(4x-9)^2(8x-18)-2(4x-9)4(4x^2-18x-16)}{(4x-9)^4} = \frac{(4x-9)(8x-18)-8(4x^2-18x-16)}{(4x-9)^3} =$
$\frac{32x^2-234x+162-32x^2+144x+128}{(4x-9)^3} = \frac{-90x+290}{(4x-9)^3}$

23) $f(x) = \frac{x^2}{2e^x+3x}$: $f'(x) = \frac{(2e^x+3x)\cdot 2x-(2e^x+3)\cdot x^2}{(2e^x+3x)^2}$, $f''(x) =$
$\frac{(2e^x+3x)^2\left[(2e^x+3x)2+(2e^x+3)(2x)-(2e^x+3)\cdot 2x-(2e^x)\cdot x^2\right]-2(e^x+3x)(2e^x+3x)\left[(2e^x+3x)\cdot 2x-(2e^x+3)\cdot x^2\right]}{(2e^x+3x)^4}$

Find the equation of the tangent line at the following points:

25) $y = e^x \sin x - 3\cos x$, $(0,-3)$: $\frac{dy}{dx} = e^x \cos x + e^x \sin x + 3\sin x$. $\left.\frac{dy}{dx}\right]_{x=0} = 1$.
Also, when $x = 0$, $y = -3$. Therefore, $y + 3 = (x - 0) \rightarrow$ $y = x - 3$.

# CHAPTER 2
# SECTION 4

## DIFFERENTIATION
## CHAIN RULE

**EXERCISES:**

Differentiate:

1) $f(x) = (5x^3 - 7x^2 + 9)^7$: $f'(x) = 7(5x^3 - 7x^2 + 9)^6(15x^2 - 14x)$

3) $y = \left(\frac{1}{2}e^x - 12x^2 - 9x + 2\right)^2$: $\frac{dy}{dx} = 2\left(\frac{1}{2}e^x - 12x^2 - 9x + 2\right)\left(\frac{1}{2}e^x - 24x - 9\right)$

5) $y = \frac{1}{\cos^2 x - cotx} = (\cos^2 x - \cot x)^{-1}$: $\frac{dy}{dx} = -(\cos^2 x - \cot x)^{-2}(-2\cos x \sin x + \csc^2 x) = -\frac{\csc^2 x - 2\cos x \sin x}{(\cos^2 x - \cot x)^2}$

7) $y = e^{5x^7 - 2x^3 + 4} + \tan^2 x - 4x^5$: $\frac{dy}{dx} = (35x^6 - 6x^2)e^{5x^7 - 2x^3 + 4} + 2\tan x \sec^2 x$

9) $f(x) = (x^2 - 9)^3(x^3 + 2x)^5$: $f'(x) = (x^2 - 9)^3 \cdot 5(x^3 + 2x)^4(3x^2 + 2) + 3(x^2 - 9)^2 \cdot 2x(x^2 - 9)^3$

11) $f(x) = (7x^2 + 2x + 8)^2(6x^3 + 2x^2 + x)^4$: $f'(x) = (7x^2 + 2x + 8)^2 \cdot 4(6x^3 + 2x^2 + x)^3(18x^2 + 4x + 1) + 2(7x^2 + 2x + 8)(14x + 2)(6x^3 + 2x^2 + x)^4$

13) $y = (\sin x + \pi x - x^2)^{-3}(3x^2 + 7x)^{\frac{3}{2}}$: $\frac{dy}{dx} = (\sin x + \pi x - x^2)^{-3} \cdot \frac{3}{2}(3x^2 + 7x)^{\frac{1}{2}}(6x + 7) - 3(\sin x + \pi x - x^2)^{-4}(\cos x + \pi - 2x)(3x^2 + 7x)^{\frac{3}{2}}$

15) $f(x) = \sqrt{e^{\frac{1}{2}x} + \tan x + x^{\frac{5}{3}}} = \left(e^{\frac{1}{2}x} + \tan x + x^{\frac{5}{3}}\right)^{\frac{1}{2}}$: $f'(x) = \frac{1}{2}\left(e^{\frac{1}{2}x} + \tan x + x^{\frac{5}{3}}\right)^{-\frac{1}{2}}\left(\frac{1}{2}e^{\frac{1}{2}x} + \sec^2 x + \frac{5}{3}x^{\frac{2}{3}}\right)$

17) $f(x) = \sqrt{\frac{5x^4 - 3x + 2}{x^2 - x}} = \left(\frac{5x^4 - 3x + 2}{x^2 - x}\right)^{\frac{1}{2}}$: $f'(x) = \frac{1}{2}\left(\frac{5x^4 - 3x + 2}{x^2 - x}\right)^{-\frac{1}{2}}\left[\frac{(x^2 - x)(20x^3 - 3) - (2x - 1)(5x^4 - 3x + 2)}{(x^2 - x)^2}\right]$

19) $y = \frac{(4x^7+3x+5)^7}{(e^{8x}-4x+6)^3}$: $\frac{dy}{dx} = \frac{(e^{8x}-4x+6)^3 \cdot 7(4x^7+3x+5)^6(28x^6+3)-3(e^{8x}-4x+6)^2(3e^{8x}-4)}{(e^{8x}-4x+6)^6}$

21) $y = \frac{(tanx+\cos^2 x-x)^2}{(5x^2-5e^x)^4}$: $\frac{dy}{dx} =$
$\frac{(5x^2-5e^x)^4 \cdot 2(tanx+\cos^2 x-x)(\sec^2 x-2\cos x\sin x-1)-4(5x^2-5e^x)^3(10x-5e^x)(tanx+\cos^2 x-x)^2}{(5x^2-5e^x)^8}$

23) $f(x) = \cos(\tan^2 x)$: $f'(x) = -\sin(\tan^2 x) \cdot 2\tan x \sec^2 x$

25) $f(t) = \cos(\cos(\cos t))$: $f'(t) = -\sin(\cos(\cos t)) \cdot (-\sin(\cos t)) \cdot (-\sin t) = -$
$\sin(\cos(\cos t)) \cdot (\sin(\cos t)) \cdot (\sin t)$

27) $y = \left(\sin\left(e^{3x^3} + 9x\right)\right)^3$: $\frac{dy}{dx} = 3\left(\sin\left(e^{3x^3} + 9x\right)\right)^2 (\cos(e^{3x^3} + 9x)(9x^2\, e^{3x^3} + 9)$

29) $s(t) = e^{\sin^2 x - \cot^2 x}$: $s'(t) = (2\sin x\cos x + 2\cot x\csc^2 x)\, e^{\sin^2 x - \cot^2 x}$

31) $f(x) = \left(3^x - e^{4x^2}\right)$: $f'(x) = 3^x \ln 3 - 8xe^{4x^2}$

Find $\frac{d^2y}{dx^2}$ for the following functions:

33) $y = (4x^7 - 3x^2)^5$: $\frac{dy}{dx} = 5(4x^7 - 3x^2)^4(28x^6 - 6x)$, $\frac{d^2y}{dx^2} = 5(4x^7 - 3x^2)^4(168x^5 -$
$6) + 20(4x^7 - 3x^2)^3(28x^6 - 6x)(28x^2 - 6x)$

35) $y = (e^x - 4x^2)^7$: $\frac{dy}{dx} = 7(e^x - 4x^2)^6(e^x - 8x)$, $\frac{d^2y}{dx^2} = 7(e^x - 4x^2)^6(e^x - 8x) +$
$42(e^x - 4x^2)^5(e^x - 8x)(e^x - 8x)$

37) $y = \sqrt{3x^3 - 7x} = (3x^3 - 7x)^{\frac{1}{2}}$: $\frac{dy}{dx} = \frac{1}{2}(3x^3 - 7x)^{-\frac{1}{2}}(9x^2 - 7)$, $\frac{d^2y}{dx^2} =$
$\frac{1}{2}(3x^3 - 7x)^{-\frac{1}{2}}(18x) - \frac{1}{4}(3x^3 - 7x)^{-\frac{3}{2}}(9x^2 - 7)(9x^2 - 7)$

Find the equation of the tangent line to the graph at the given point:

39) $y = \sin^2 x - e^{2x}$, $(0, -1)$: $\frac{dy}{dx} = 2\sin x\cos x - 2e^{2x}$, $\left.\frac{dy}{dx}\right]_{x=0} = -2 \to \quad y + 1 =$
$-2(x - 0) \to \quad y = -2x - 1$

41) $f(x) = \sin(\sin x)$, $(0,0)$: $\frac{dy}{dx} = \cos(\sin x) \cdot \cos x$, $\left.\frac{dy}{dx}\right]_{x=0} = 1 \rightarrow \quad y - 0 = 1(x - 0) \rightarrow$

$y = x$

**CHAPTER 2**
**SECTION 5**

**DIFFERENTIATION**
**IMPLICIT DIFFERENTIATION**

**EXERCISES:**

Find $\frac{dy}{dx}$ by using Implicit Differentiation:

1) $x^2 + y^2 = 9$: $2x + 2y\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{y}{x}$

3) $4x^2 - 2y = 3x$: $8x - 2\frac{dy}{dx} = 3 \rightarrow -2\frac{dy}{dx} = \frac{3}{8x} \rightarrow \frac{dy}{dx} = -\frac{3}{16x}$

5) $3x^3y^2 - 4xy = 2y^3$: $3x^3 \cdot 2y\frac{dy}{dx} + 9x^2y^2 - 4x \cdot \frac{dy}{dx} - 4y = 6y^2\frac{dy}{dx} \rightarrow 6x^3y\frac{dy}{dx} - 4x\frac{dy}{dx} - 6y^2\frac{dy}{dx} = 4y - 9x^2y^2 \rightarrow \frac{dy}{dx} = \frac{4y-9x^2y^2}{6x^3y-4x-6y^2}$

7) $4ye^x - \sin x = \tan^2 y$: $4ye^x + 4 \cdot \frac{dy}{dx}e^x - \cos x = 2\tan y \cdot \sec^2 y\frac{dy}{dx} \rightarrow 4\,e^x\frac{dy}{dx} - 2\tan y \sec^2 y\frac{dy}{dx} = \cos x - 4ye^x \rightarrow \frac{dy}{dx} = \frac{\cos x - 4ye^x}{4\,e^x - 2\tan y \sec^2 y}$

9) $e^{2xy} - \cos y = \sin(xy)$: $e^{2xy} \cdot \left(2x\frac{dy}{dx} + 2y\right) + \sin y\frac{dy}{dx} = \cos(xy) \cdot \left(x\frac{dy}{dx} + y\right) \rightarrow 2x\,e^{2xy}\frac{dy}{dx} + 2y\,e^{2xy} + \sin y\frac{dy}{dx} = x\cos(xy)\frac{dy}{dx} + y\cos(xy) \rightarrow 2x\,e^{2xy}\frac{dy}{dx} + \sin y\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} = y\cos(xy) - 2y\,e^{2xy} \rightarrow \frac{dy}{dx} = \frac{y\cos(xy) - 2y\,e^{2xy}}{2x\,e^{2xy} + \sin y - x\cos(xy)}$

11) $(4xy - \tan(2xy) + e^y)^{=}1$: $(4x\frac{dy}{dx} + 4y - \sec^2\ (2xy) \cdot \left(2x\frac{dy}{dx} + 2y\right) + e^y) = 0 \rightarrow (4x\frac{dy}{dx} + 4y - 2x\sec^2(2xy)\frac{dy}{dx} - 2y\sec^2(2xy) + e^y) \rightarrow \frac{dy}{dx} = \frac{4x - 2x\sec^2(2xy)}{2y\sec^2(2xy) - 4y - e^y}$

13) $\tan(2xy^2) + 4e^{x^2y} = 4y^7$: $\sec^2(2xy^2) \cdot \left(4xy\frac{dy}{dx} + 2y^2\right) + 4\left(x^2\frac{dy}{dx} + 2xy\right)e^{x^2y} = 28y^2\frac{dy}{dx} \rightarrow 4xy\sec^2(2xy^2)\frac{dy}{dx} + 2y^2\sec^2(2xy^2) + 4x^2e^{x^2y}\frac{dy}{dx} + 8xye^{x^2y} = 28y^2\frac{dy}{dx} \rightarrow \frac{dy}{dx} = -\frac{4xy\,sec^2(2xy^2) + 4x^2e^{x^2y} - 28y^2}{2y^2\sec^2(2xy^2) + 8xye^{x^2y}}$

Find $\frac{dy}{dx}$ at given points:

15) $\sin(xy) - e^x = \frac{1}{2}y$ at the point (0,-2):

$$\cos(xy) \cdot \left(x\frac{dy}{dx} + y\right) - e^x = \frac{1}{2}\frac{dy}{dx}$$

$$\rightarrow \quad x\cos(xy)\frac{dy}{dx} + y\cos(xy) - e^x = \frac{1}{2}\frac{dy}{dx} \rightarrow \quad 0 - 2(1) - 1 = \frac{1}{2}\frac{dy}{dx} \rightarrow \quad \frac{dy}{dx} = -6$$

17) Find the equation of the tangent line to the graphs in Exercises 14)-16) at the given points in Exercises 14)-16). (We will do number 15), since it is odd). $\frac{dy}{dx} = -6, \ \ (x, y) = (0, -2) \rightarrow$ $y + 2 = -6(x - 0) \rightarrow \quad y = -6x - 2$

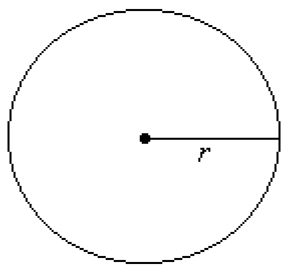
**CHAPTER 2**
**SECTION 6**

**DIFFERENTIATION**
**RELATED RATES**

1) Let $y = 2x^2y - 3y$. Find $\frac{dy}{dt}$ when $\frac{dx}{dt} = 2$ and $x = 1, y = 0$: $\frac{dy}{dt} = 2x^2\frac{dy}{dt} + 4xy\frac{dx}{dt} - 3\frac{dy}{dt} \rightarrow \frac{dy}{dt}(1 - 2 \cdot 1^2 + 3) = 4 \cdot 1 \cdot 0 \rightarrow \frac{dy}{dt} = 0$

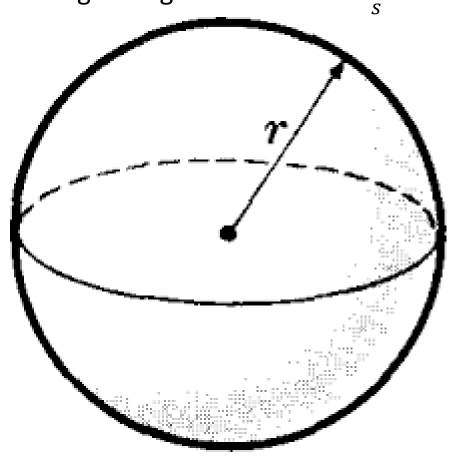
3) Let $e^x = xy^2 - cosz + 2x$. Find $\frac{dz}{dt}$ when $\frac{dx}{dt} = 0, \frac{dy}{dt} = 1, x = 0, y = 3, z = \frac{\pi}{2}$: $e^x\frac{dx}{dt} = 2xy\frac{dy}{dt} + y^2\frac{dx}{dt} + \sin z\frac{dz}{dt} + 2\frac{dx}{dt} \rightarrow 0 = 0 + 1 \cdot \frac{dz}{dt} + 0 \rightarrow \frac{dy}{dt} = 0$

5) A raindrop keeps growing larger as it spreads out on the concrete. It is growing uniformly, and is perfectly circular. What is the rate of increase of its area, when its radius is $2mm$? Its radius is increasing at a rate of $1\frac{mm}{s}$.



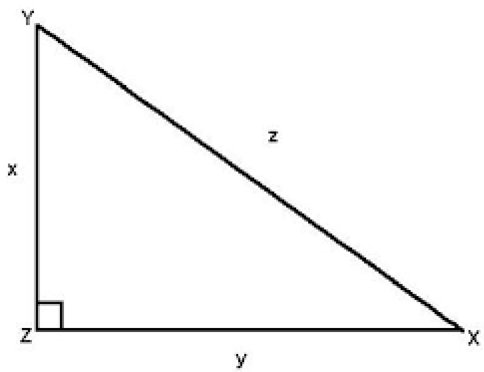

$A = \pi r^2 \rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt} \rightarrow \frac{dA}{dt} = 2\pi \cdot 2 \cdot 1 = 4\pi$

7) Sally is making a snowball. It is perfectly spherical. Its volume is growing at a rate of $12\frac{cm^3}{s}$. At what rate is its radius increasing when its radius is $20\ cm$?




$$V = \frac{4}{3}\pi r^3 \rightarrow \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \rightarrow 12 = 4\pi 20^2\frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{3}{400\pi}$$

9) Two cars are headed away from each other. Car A is headed North, and car B is headed East. Car A is traveling at 55 mph, and car B is traveling at 75 mph. At what rate is their distance increasing two hours after they leave each other?




We let Z be the starting point for both cars. Let y be the distance from car B away from Z. Let x be the distance from car A away from Z. We need a formula. We use the Pythagorean Theorem:

$x^2 + y^2 = z^2$

We note that $\frac{dx}{dt} = 55 \text{ mph}$, and $\frac{dy}{dt} = 75 \text{ mph}$. Let's take time derivatives:

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2z \cdot \frac{dz}{dt}$$

Next, we need to calculate $x, y$ and $z$:

At 2 hours after leaving, $x = 110 \text{ miles}$ , $y = 150 \text{ miles}$, we get: $110^2 + 150^2 = z^2 \rightarrow \quad z \approx 186$

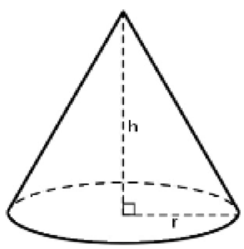
Now we substitute all known values:

$$2 \cdot 110 \cdot (55) + 2 \cdot 150 \cdot (75) = 2 \cdot 186 \cdot \frac{dz}{dt} \rightarrow \qquad \frac{dz}{dt} \approx 67.77 \text{ mph}$$

11) A rectangular ice cube is melting. Its height is shrinking at a rate of 4 $\frac{mm}{s}$. Its width is shrinking at a rate of 5 $\frac{mm}{s}$, and its volume is decreasing at a rate of 9 $\frac{mm^3}{.s}$. At what rate is depth shrinking, when its height is 6 $cm$, its width is 3 $cm$, and its depth is 1.5 $cm$?

Let $x$ represent depth, $y$ represent width, and $z$ represent height. $V = xyz \rightarrow \frac{dV}{dt} = xy\frac{dz}{dt} + xz\frac{dy}{dt} + yz\frac{dx}{dt} \rightarrow\ 9 = 1.5 \cdot 3 \cdot 4 + 1.5 \cdot 6 \cdot 5 + 3 \cdot 6 \cdot \frac{dx}{dt} \rightarrow\ 9 = 18 + 45 + 18 \cdot \frac{dx}{dt} \rightarrow \frac{dx}{dt} = 4\frac{mm}{s}$

13) Coal is being dumped into a pile of in the shape of a cone. The radius is increasing at a rate of 7 m/s, and the height is increasing at a rate of 10 m/s. How fast is the surface area increasing when the radius is 5 meters, and the height is 10 meters?

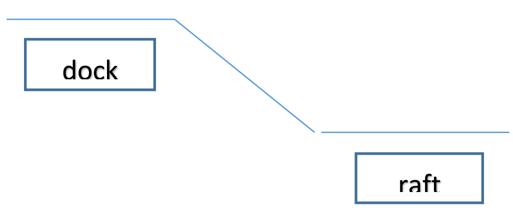


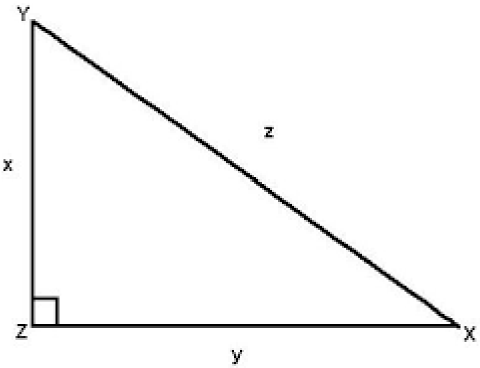

$$S = \pi r^2 + \pi r\sqrt{h^2 + r^2} = \pi r^2 + \pi\sqrt{r^2h^2 + r^4} \rightarrow$$

$$\frac{dS}{dt} = 2\pi r\frac{dr}{dt} + \frac{\pi\left(2r^2h\frac{dh}{dt} + 2rh^2\frac{dr}{dt} + 4r^3\frac{dr}{dt}\right)}{\sqrt{r^2h^2 + r^4}}$$

$$= \frac{\pi(2 \cdot 5^2 \cdot 10 \cdot 10 + 2 \cdot 5 \cdot 10^2 \cdot 7 + 4 \cdot 5^3 \cdot 7)}{\sqrt{5^2 \cdot 10^2 + 5^4}} \approx \frac{38138.9}{55.9} \approx 682.27$$

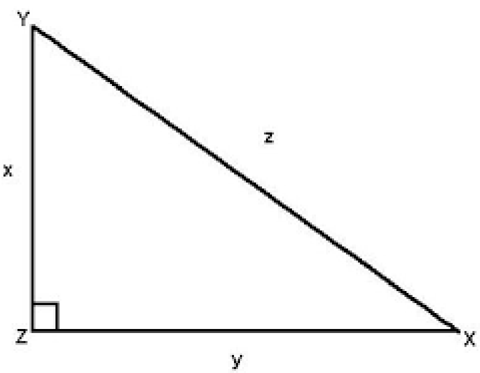
15) A flat raft is moving away from a dock. The dock is 2 m higher than the raft. If the raft is moving at a rate of 3 $\frac{m}{s}$, what is the rate it is moving away from the dock, when it is 6 m away?





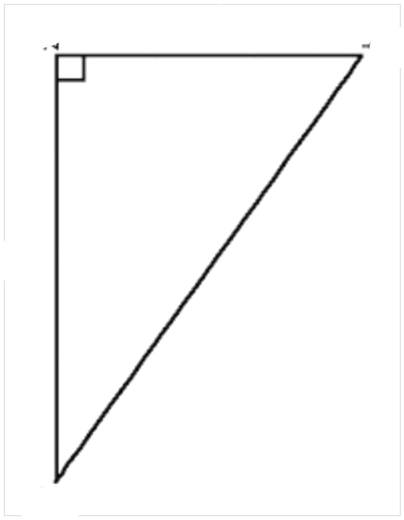

$$x^2 + y^2 = z^2 \rightarrow \quad 2^2 + y^2 = 6^2 \rightarrow \quad y = \sqrt{32} = 4\sqrt{2} \rightarrow \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt} \rightarrow \quad 2 \cdot 2 \cdot 0 + 2 \cdot 4\sqrt{2} \cdot 3 = 2 \cdot 4\sqrt{2} \cdot \frac{dz}{dt} \rightarrow \quad \frac{dz}{dt} = \frac{24\sqrt{2}}{8\sqrt{2}} = 3\frac{m}{s}.$$

17) Two people are heading toward each other. If one is walking south at 3 mph, and the other is running west at 7 mph. At what rate are the two approaching each other when the runner is 5 miles from the destination, and the walker is 12 miles from the destination?




$$x^2 + y^2 = z^2, \;\; z = 13, \;\; 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt} \rightarrow \quad 2 \cdot 12 \cdot 3 + 2 \cdot 7 \cdot 5 = 2 \cdot 13 \cdot \frac{dz}{dt} \rightarrow \frac{dz}{dt} = \frac{142}{26} \approx 5.46\, mph$$

19) A plane is flying at an altitude 6.5 miles. It will pass over a radar station. If its rate from the plane to the station is decreasing at a rate of 550 mph when $z$ is 20 miles, what is the speed of the plane?



Let $x =$ horizontal distance, $y =$ vertical distance, and $z =$ hypotenuse. If $z$ is decreasing at a rate of 550 mph when $z$ is 20 miles, what is the speed of the plane? $y = 6.5$ miles. $z = 20$ miles. Let us find $x$. $x^2 + 6.5^2 = 20^2 \rightarrow \quad x = \sqrt{400 - 42.25} \approx 18.91.$

Our equation is: $x^2 + 6.5^2 = z^2$

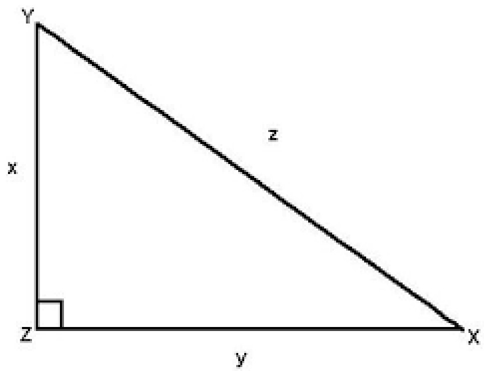
Taking time derivatives:

$$2x \cdot \frac{dx}{dt} + 0 = 2z \cdot \frac{dz}{dt}$$

Substituting all knowns:

$$2 \cdot 18.91 \cdot \frac{dx}{dt} = 2 \cdot 20 \cdot 550 \rightarrow \quad \frac{dx}{dt} = 581.7\ mph$$

21) A right triangle has its length decreasing at a rate of 1 $\frac{\text{cm}}{s}$. How fast is the angle decreasing when its length is 4 and its width is 3 if the base is fixed?.




$$\tan\theta = \frac{x}{y} \rightarrow \quad \theta \approx .64, \quad \sec^2\theta \frac{d\theta}{dt} = \frac{1}{y}\frac{dx}{dt} - \frac{x}{y^2}\frac{dy}{dt}$$

$$\rightarrow \quad \sec^2 .64 \frac{d\theta}{dt} = \frac{1}{4}\cdot 1 - \frac{3}{16}\cdot 0 \rightarrow \quad .8\frac{d\theta}{dt} = \frac{1}{4} \rightarrow \quad \frac{d\theta}{dt} \approx .2\ radians$$

**CHAPTER 2**
**SECTION 7**

**DIFFERENTIATION**
**DERIVATIVES OF LOGARITHMIC FUNCTIONS AND INVERSE TRIGONOMETRIC FUNCTIONS**

**EXERCISES:**

Differentiate:

1) $f(x) = \ln x^2 = 2\ln x$: $f'(x) = \frac{2}{x}$

3) $y = 3x^2 \ln(3x)$: $\frac{dy}{dx} = 3x^2 \cdot \frac{1}{x} + 6x\ln(3x) = 3x + 6x\ln(3x)$

5) $y = \ln(4x^3 - x^{-2})$: $\frac{dy}{dx} = \frac{12x^2 + x^{-3}}{4x^3 - x^{-2}}$

7) $y = \ln(e^x) = x$: $\frac{dy}{dx} = 1$

9) $f(t) = ln[(2x+7)^2(3x^4+3x^2)^3] = 2\ln(2x+7) + 3\ln(3x^4+3x^2)$: $f'(t) = \frac{4}{2x+7} + \frac{36x^3+18x}{3x^4+3x^2}$

11) $f(x) = ln\left[\frac{x+1}{2x-2}\right] = \ln(x+1) - \ln(2x-2)$: $f'(x) = \frac{1}{x+1} - \frac{2}{2x-2}$

13) $f(x) = ln\left[\frac{\sqrt{2x+3}}{(4x^2-3x)^2}\right] = \frac{1}{2}\ln(2x+3) - 2\ln(4x^2-3x)$: $f'(x) = \frac{1}{2x+3} - \frac{16x-3}{4x^2-3x}$

15) $y = \ln(\ln(\ln(x^2)))$: $\frac{dy}{dx} = \frac{1}{(\ln(\ln(x^2)))} \cdot \frac{1}{(\ln(x^2))} \cdot \frac{2}{x}$

17) $y = ln\sqrt{(\tan^2 x - 5x)} = \frac{1}{2}\ln(\tan^2 x - 5x)$: $\frac{dy}{dx} = \frac{2\tan x \sec x - 5}{2\tan^2 x - 10x}$

19) $y = \cos^{-1}(3x^2 - 2)$: $\frac{dy}{dx} = -\frac{1}{\sqrt{1-(3x^2-2)^2}} \cdot 6x = -\frac{6x}{\sqrt{-9x^4+12x^2-3}}$

21) $f(t) = \frac{\operatorname{arccot}(2x^2)}{\ln(3x)}$: $f'(t) = \frac{\ln(3x)\cdot\frac{1}{\sqrt{1+4x^4}}\cdot 4x - \frac{1}{x}\cdot\operatorname{arccot}(2x^2)}{(\ln(3x))^2} = \frac{1}{(\ln(3x))^2}\left(\frac{4x\ln(3x)}{\sqrt{1+4x^4}} - \frac{\operatorname{arccot}(2x^2)}{x}\right)$

23) $y = \arcsin(14x^2 - 12x)^{\frac{1}{3}}$: $\frac{dy}{dx} = \frac{28x-12}{\sqrt{1-(14x^2-12x)^{\left(\frac{2}{3}\right)}}}$

25) $f(x) = \arcsin(\arcsin x)$: $f'(x) = \frac{1}{\sqrt{(1-\arcsin^2 x)}} \cdot \frac{1}{\sqrt{1-x^2}}$

Find $\frac{dy}{dx}$ for the following functions using Implicit Differentiation:

27) $\ln y = \arctan 2x + y^2 \ln x$: $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{1+4x^2} + \frac{y^2}{x} + 2y \ln x \frac{dy}{dx} \rightarrow \quad \frac{dy}{dx} = \frac{\frac{2}{1+4x^2} + \frac{y^2}{x}}{\frac{1}{y} - 2y \ln x}$

Use Logarithmic Differentiation to find the derivative of the following functions:

29) $y = \sqrt{x^2 - \ln(3x)}\,(4x^3+4)^3 \rightarrow \quad \ln y = \ln \sqrt{x^2 - \ln(3x)}\,(4x^3+4)^3 \rightarrow$

$\ln y = \frac{1}{2}\ln(x^2 - \ln(3x)) + 3\ln(4x^3+4) \rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{2x - \frac{1}{x}}{2(x^2 - \ln(3x)} + \frac{36x^2}{4x^3+4} \rightarrow \quad \frac{dy}{dx} =$
$\left[\sqrt{x^2 - \ln(3x)}\,(4x^3+4)^3\right]\left[\frac{2x-\frac{1}{x}}{2(x^2-\ln(3x)} + \frac{36x^2}{4x^3+4}\right]$

31) $f(x) = \frac{(x^4-20x)^2}{\sqrt{4x+10}} \rightarrow \quad \ln y = \ln \frac{(x^4-20x)^2}{\sqrt{4x+10}} \rightarrow \quad \ln y = 2\ln(x^4 - 20x) \quad - \frac{1}{2}\ln(4x+10) \rightarrow$
$\frac{1}{y}\frac{dy}{dx} = \frac{8x^3-40}{x^4-20x} - \frac{4}{8x+20} \rightarrow \quad \frac{dy}{dx} = \frac{(x^4-20x)^2}{\sqrt{4x+10}} \cdot \left(\frac{8x^3-40}{x^4-20x} - \frac{4}{8x+20}\right)$

33) $y = \frac{(3x^2-7x)^2(4x-9)^3}{(20x^3+2)^4\sqrt{14x^2-8}} \rightarrow \quad \ln y = 2\ln(3x^2-7x) + 3\ln\big(4x-9) - 4\ln(20x^3+2) -$
$\frac{1}{2}\ln(14x^2-8)\big) \rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{12x-14}{3x^2-7x} + \frac{12}{4x-9} - \frac{240x^2}{80x^3-8} - \frac{28x}{28x^2-16} \rightarrow \quad \frac{dy}{dx} =$
$\frac{(3x^2-7x)^2(4x-9)^3}{(20x^3+2)^4\sqrt{14x^2-8}} \cdot \left(\frac{12x-14}{3x^2-7x} + \frac{12}{4x-9} - \frac{240x^2}{80x^3-8} - \frac{28x}{28x^2-16}\right)$

35) $y = x^{e^x} \rightarrow \quad \ln y = e^x \ln x \rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{e^x}{x} + e^x \ln x \rightarrow \quad \frac{dy}{dx} = x^{e^x}\left(\frac{e^x}{x} + e^x \ln x\right)$

## CHAPTER 2
## SECTION 8
## LINEAR APPROXIMATIONS AND DIFFERENTIALS
## HYPERBOLIC FUNCTIONS

**EXERCISES:**

Find the Linear approximation for the following functions at the given value for $x$:

1) $f(x) = e^x + 2x^2,\ x = 0$: $L(x) = f(a) + f'(a)(x - a),\quad f'(x) = e^x + 4x,\quad f'(0) = 1 \rightarrow \quad L(x) = 1 + 1(x - 0) \rightarrow \quad L(x) = x + 1$

3) $f(x) = 3x^3 - x^{\frac{1}{3}},\quad x = 1$: $f'(x) = 9x^2 - \frac{1}{3}x^{-\frac{2}{3}},\quad f'(1) = \frac{26}{3},\quad L(x) = 2 + \frac{26}{3}(x - 1) \rightarrow$ $L(x) = \frac{26}{3}x - \frac{20}{3}$

Find the differential:

5) $y = \sin^2 x \cos x \rightarrow \quad dy = (-\sin^2 x \cdot \sin x + 2\sin x \cos x \cdot \cos x)\, dx \rightarrow \quad dy = (2\sin x \cos^2 x - \sin^3 x)\, dx$

7) $y = \ln(x^2) - \tan^{-1} x \rightarrow \quad dy = \left(\frac{2}{x} - \frac{1}{1+x^2}\right) dx$

9) $y = \ln(\cos x \cdot \tan x) \rightarrow \quad dy = \frac{\cos x \sec^2 x - \sin x \tan x}{\cos x \tan x}\, dx \rightarrow$
$dy = \left(\frac{1}{\cos x \sin x} - \tan x\right) dx \rightarrow \quad dy = (\sec x \tan x - \tan x)\, dx$

Find the differential, $dy$, for the given values of $x$ and $dx$:

11) $y = 3x^2 - 6x,\quad x = 2,\ dx = 0.1$: $dy = (6x - 6)\, dx \rightarrow \quad dy = (6 \cdot 2 - 6) \cdot 0.1 \rightarrow$ $dy = .6$

13) $y = \sin \pi x,\quad x = \frac{1}{2},\ dx = 0.01$: $dy = \pi \cos \pi x\, dx \rightarrow \quad dy = \pi \cos \frac{\pi}{2} \cdot 0.01 \rightarrow \quad dy = .0\pi$

Compare $\Delta y$ and $dy$ for the following functions at the given values for $x$ and $\Delta x = dx$: (For 15) and 16)):

15) $y = x^3 - 2,\ x = 3,\ \Delta x = 0.1$: $dy = 3x^2\, dx \rightarrow \quad dy = 3 \cdot 3^2 \cdot 0.1 = 2.7$: $\Delta y = 3.1^3 - 2 - (3^3 - 2) = 2.791$

17) A circular disk has a radius of 3 cm, and a possible error of $0.02\ cm$. Use differentials to estimate the maximum error in its Area.

$A = \pi r^2 \rightarrow \quad dA = 2\pi r\ dr \rightarrow \quad dA = 2\pi \cdot 3 \cdot 0.02 = .12\pi\ cm^2$

19) A cube with sides 5 cm, has a possible error of 0.1 cm. Use differentials to estimate the maximum error in its Surface Area.

$S = 6x^2 \rightarrow \quad dS = 12x\ dx \rightarrow \quad dS = 12 \cdot 5 \cdot 0.1 = 6\ cm^2$

**CHAPTER 3**
**SECTION 1**
**MAXIMUM AND MINIMUM VALUES (EXTREMA)**

**EXERCISES:**

Find all critical numbers for the following functions:

1) $f(x) = 4x^2 - 12 \rightarrow \quad f'(x) = 8x\colon\ 8x = 0 \rightarrow \quad x = 0$

3) $y = 3x^3 - 27x^2 \rightarrow \ \frac{dy}{dx} = 9x^2 - 54x = 0 \rightarrow \quad 9x(x-6) = 0 \rightarrow \quad x = 0{,}6$

5) $y = e^x - x \rightarrow \ \frac{dy}{dx} = e^x - 1 = 0 \rightarrow \quad x = 0$

7) $f(t) = 14t - 2 \rightarrow \quad f'(t) = 14$, so not critical numbers

9) $y = \tan^2 x \rightarrow \ \frac{dy}{dx} = 2\tan x \sec^2 x = 0 \rightarrow \quad \sin x = 0 \rightarrow \quad x = n\pi,\ \ n = 0,1,2, \ldots$

Find the Absolute Maximum and the Absolute Minimum for the following functions over the given interval:

11) $f(x) = 2x^2 - 20$, [-2,2]: $f'(x) = 4x = 0 \rightarrow \quad x = 0;\ f(-2) = f(2) = -12,\ f(0) = -20$. Therefore, $(-2,-12)$, and $(2,-12)$ is absolute max, and $(0,-20)$ is absolute minimum.

13) $f(x) = 3x^2 - 5x - 17,\ [0,2]\colon\ f'(x) = 6x - 5 = 0 \rightarrow \quad x = \frac{5}{6};\ \ f(0) = -17,\ f\left(\frac{5}{6}\right) = -\frac{229}{12}$. Therefore, $(0,-17)$ is absolute max, and $\left(0,-\frac{229}{12}\right)$ is absolute minimum.

15) $f(x) = -x^3 + 3x + 1,\ [-2,2]\colon\ f'(x) = -3x^2 + 3 = 0 \rightarrow \quad x = \pm 1;\ \ f(-2) = 3,\ f(-1) = -1,\ f(1) = 3,\ f(2) = -1$. Therefore, $(1,3), (-2,3)$ are absolute max, and $(-1,-1), (2,-1)$ are absolute minima.

17) $f(x) = x^3 - 6x^2 + 9x - 2,\ [-4,1]\colon\ f'(x) = 3x^2 - 12x + 9 = 0 \rightarrow \quad x^2 - 4x + 3 = 0 \rightarrow x = 1{,}3;\ \ f(-4) = -198,\ f(1) = 2,\ f(3)$ is outside our interval. Therefore, $(1,2)$ is absolute max, and $(-4,-198)$ is absolute minimum.

19) $f(x) = x^4 - 3x^3,\ [-1,4]\colon\ f'(x) = 4x^3 - 9x^2 = 0 \rightarrow \quad x^2(4x-9) = 0 \rightarrow \quad x = 0, \frac{9}{4};\ \ f(-1) = 4,\ f(0) = 0,\ f\left(\frac{9}{4}\right) = -\frac{2187}{256},\ f(4) = 64$. Therefore, $(4,64)$ is absolute max, and $\left(\frac{9}{4}, -\frac{2187}{256}\right)$ is absolute minimum.

21) $f(\theta) = 2\cos\theta - \theta$, $[0, \pi]$: $f'(\theta) = -2\sin\theta - 1 = 0 \rightarrow \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ (not in our interval); $f(0) = 2$, $f(\pi) = -2 - \pi$. Therefore, $(0,2)$ is absolute max, and $(\pi, -2-\pi)$ is absolute minimum

23) $f(x) = \frac{2}{x^2+2}$, $[-2,3]$: $f'(x) = -\frac{4x}{(x^2+2)^2} = 0 \rightarrow \quad x = 0$, (Note: denominator never equals 0); $f(-2) = \frac{1}{3}$, $f(0) = 1$, $f(3) = \frac{2}{11}$. Therefore, $(0,1)$ is absolute max, and $\left(3, \frac{2}{11}\right)$ is absolute minimum

25) $f(x) = (x+1)^{\frac{2}{3}}$, $[-2,2]$: $f'(x) = \frac{2}{3(x+1)^{\frac{1}{3}}} = 0 \rightarrow \quad 2 = 0$ (no solutions). But, $x = -1$ is a value such that $f'(x)$ does not exist. $f(-2) = 1$, $f(-1) = 0$, $f(2) = \sqrt[3]{9}$. Therefore, $(2, \sqrt[3]{9})$ is absolute max, and $(-1,0)$ is absolute minimum.

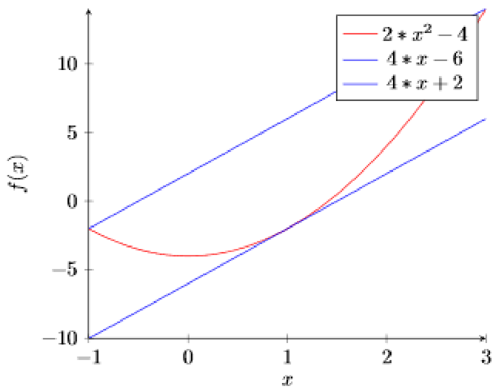
27) $f(x) = -3(x-1)^{\frac{2}{3}} + 2$, $[0,2]$: $f'(x) = -\frac{2}{(x-1)^{\frac{1}{3}}} = 0 \rightarrow \; -2 = 0$ (no solutions). But, $x = 1$ is a value such that $f'(x)$ does not exist. $f(0) = -1$, $f(1) = 2$, $f(2) = -1$. Therefore, $(1,2)$ is absolute max, and $(0,-1)$, $(2,-1)$ are absolute minima.

# CHAPTER 3
## SECTION 2
## THE MEAN VALUE THEOREM

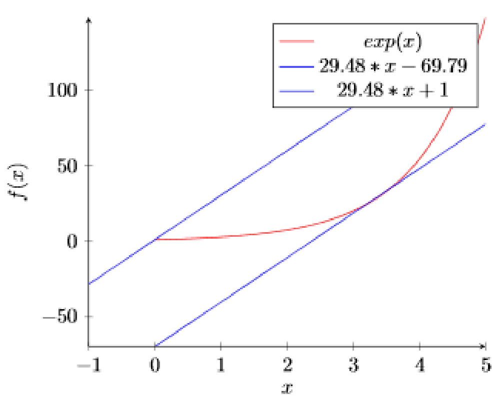
**EXERCISES:**

Sketch the graph of the following functions over the given interval. Sketch the slope of the secant line over the interval, then sketch the tangent lines with the same slope as the secant line: (You can use a graphing calculator if needed).

1) $f(x) = 2x^2 - 4$, $[-1,3]$: $f'(x) = 4x = \frac{f(3)-f(-1)}{3+1} = \frac{14+2}{4} = 4 \rightarrow$ $x = 1 \rightarrow$ $f'(1) = 4$; $f(1) = -2 \rightarrow$ equation of tangent line is $y + 2 = 4(x-1) \rightarrow$ $y = 4x - 6$, and equation of the secant line is $y + 2 = 4(x+1) \rightarrow$ $y = 4x + 2$:




3) $f(x) = e^x, [0,5]$: $f'(x) = e^x = \frac{e^5-e^0}{5-0} = \frac{e^5-1}{5} \approx 29.48 \rightarrow$ $x \approx \ln 29.48 \approx 3.38 \rightarrow$ $f'(3.38) \approx e^{3.38} \approx 29.48 \rightarrow$ equation of the tangent line is $y - 29.48 = 29.48(x - 3.38) \rightarrow$ $y = 29.48x - 69.79$, and equation of the secant line is $y - 1 = 29.48(x - 0) \rightarrow$ $y = 29.48x + 1$



Verify that the function satisfies the 3 conditions for Rolle's Theorem over the given interval. Then find all values $c$ that satisfy its conclusion.

5) $f(x) = x^2 - 9,\ [-2,2]$: We will use Rolle's Theorem to find all numbers $c$ that satisfy the conclusion of Rolle's Theorem. We start by showing this function satisfies the 3 conditions of Rolle's Theorem:

1) $f$ is continuous on $[-2,2]$. True, polynomials are continuous everywhere.
2) $f$ is differentiable on $(-2,2)$. True, polynomials are differentiable everywhere.
3) $f(-2) = -5,\ f(2) = -5$. This condition holds true.

Then, there exists a $c$ such that $f'(c) = 0$ in $(a, b)$. $f'(x) = 2x = 0 \rightarrow \quad x = 0$. 0 is in the open interval $(-2,2)$. This is only value for $c$ that we obtain.

7) $f(x) = \sin 2x,\ \ [0, \pi]$: We will use Rolle's Theorem to find all numbers $c$ that satisfy the conclusion of Rolle's Theorem. We start by showing this function satisfies the 3 conditions of Rolle's Theorem:

1) $f$ is continuous on $[0, \pi]$. True, polynomials are continuous everywhere.
2) $f$ is differentiable on $(0, \pi)$. True, polynomials are differentiable everywhere.
3) $f(0) = 0,\ f(\pi) = 0$. This condition holds true.

Then, there exists a $c$ such that $f'(c) = 0$ in $(a, b)$. $f'(x) = 2\cos 2x = 0 \rightarrow \quad x = \frac{\pi}{4}$. 0 is in the open interval $(0, \pi)$. This is only value for $c$ that we obtain.

Verify that the function satisfies the 2 conditions for the Mean Value Theorem over the given interval. Then find all values $c$ that satisfy its conclusion:

9) $f(x) = 3x^2 + 2x - 3,\ \ [0,1]$: 1) Continuous on $[0,1]$, 2) Differentiable on $(0,1)$: $f'(x) = 6x + 2 = \frac{f(1)-f(0)}{1-0} = 5 \rightarrow \quad x = \frac{1}{2}$

11) $f(x) = 3\ln 2x,\ \ [1,2]$: 1) Continuous on $[1,2]$, 2) Differentiable on $(1,2)$: $f'(x) = \frac{3}{x} = \frac{f(2)-f(1)}{2-1} = \ln 64 - \ln 8 = \ln 8 \rightarrow \quad x = \frac{3}{\ln 8} \approx 1.44$

13) You are skiing down a mountain with an average speed of 20 mph. Your distance vs time is modeled by the function $f(t) = \frac{1}{3}t^3 - \frac{7}{3}t$. You skied for 8 minutes. . How many minutes after you began your descent, did you go 20 mph?

$f(t)$ continuous and differentiable everywhere. $f'(t) = t^2 - \frac{7}{3} = \frac{1}{3}\frac{miles}{minute} \rightarrow \quad t^2 = \frac{8}{3} \rightarrow$

$t = \frac{2\sqrt{2}}{3}\ minutes$

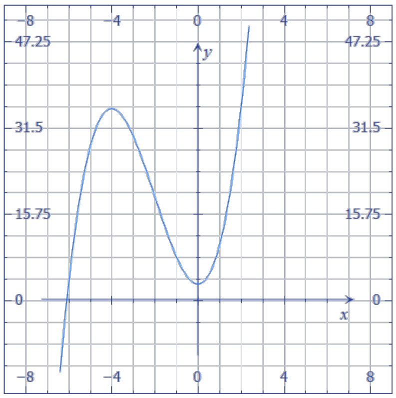
15) You are traveling in car. You travel for 90 miles in 2 hours. Prove that your speed was 45 mph at least once, and at what hour after leaving it occurred? (Your model function is continuous and differentiable everywhere).

Your average speed is $\frac{90\ miles}{2\ hours} = 45\ mph$. Therefore, by the Mean Value Theorem, your instantaneous speed had to be 45 miles per hour at least once.

**CHAPTER 3**
**SECTION 3**
**HOW DERIVATIVES AFFECT THE SHAPE OF A GRAPH**

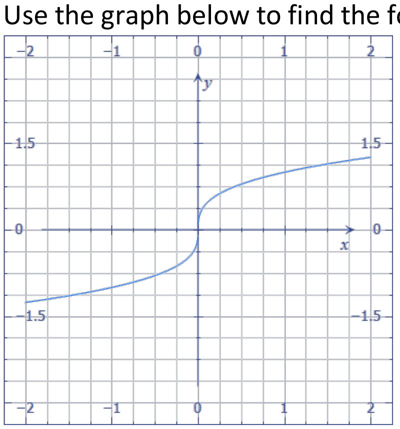
**EXERCISES:**

1) Use the graph below to find the following:




a) The location of any local maximum and local minimum values: Give $x$-values, and you can approximate the $y$-values. Solution: Local maximum at (-4,34), and local minimum at (0,3)
b) The intervals for which $f$ is increasing, and where $f$ is decreasing. Solution: $f$ is increasing from $(-\infty,-4) \cup (0,\infty)$. $f$ is decreasing from $(-4,0)$.
c) Approximate any inflection points. Inflection point at $(-2,15.75)$.
d) Find all intervals of concavity. Concave down from $(-\infty,-2)$, and concave up from $(-2,\infty)$.

3) Use the graph below to find the following:

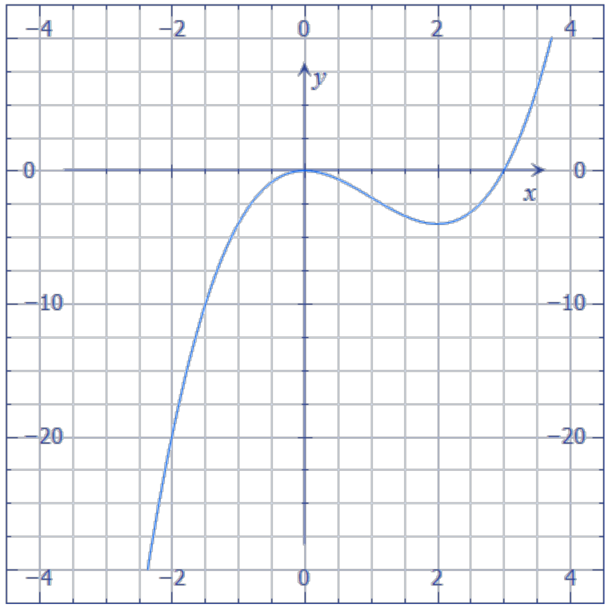



a) The location of any local maximum and local minimum values: Solution: None.
b) The intervals for which $f$ is increasing, and where $f$ is decreasing. Solution: $f$ is increasing from $(-\infty,0) \cup (0,\infty)$
c) Find any inflection points. Inflection point at $(0,0)$

d) Find all intervals of concavity. $f$ is concave up from $(-\infty, 0)$ and concave down from $(0, \infty)$.

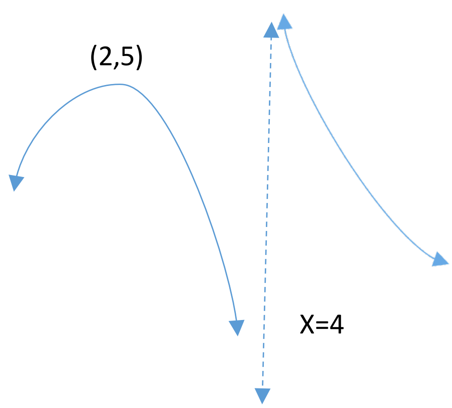
Use the following information to sketch a graph that meets the conditions given. Note that your graph can be correct without being unique:

5) $f$ is increasing and concave up on $(-\infty, 1)$ and increasing and concave down on $(1, \infty)$.

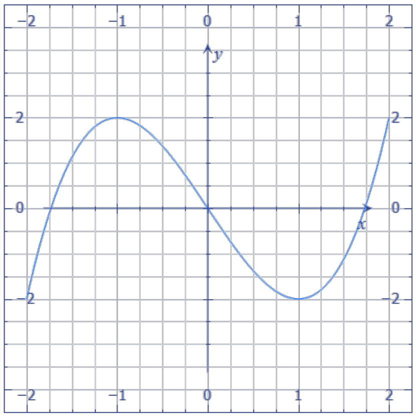



(Note: This is not the only possible graph)

7) $f$ has a local maximum at $(2{,}5)$, is concave down from $(-\infty, 4)$, has a vertical asymptote at $x = 4$, and concave up from $(4, \infty)$.




9) $f(0) = 0,\ f(1) = -2,\ f(-1) = 2.$ $f$ has a local minimum at $(1, -2)$, a local maximum at $(-1{,}2)$, an infection point at $(0{,}0)$.




For the functions given, find the following:

a) The intervals for which $f$ is increasing or decreasing
b) The local maximum and minimum values of $f$
c) The inflection point(s) and intervals of concavity
d) Use steps a)-c) to sketch the graph of the function:

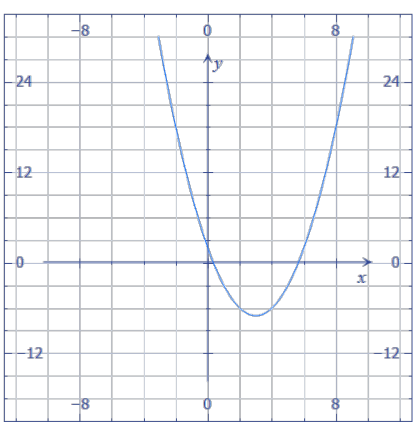
11) $f(x) = x^2 - 6x + 2$:

a) Let us first find all critical points of $f$: $f'(x) = 2x - 6 = 0 \rightarrow \quad x = 3$. So $(3, -7)$ is our only critical point. $f'(0) = -6, \;\; f'(4) = 8$. Therefore, $f$ is decreasing from $(-\infty, 3)$ and increasing from $(3, \infty)$.

b) Since $f$ is decreasing before $x = 3$, and increasing after, $f$ has a local minimum at (3,-7).

c) To find inflection point(s): $f''(x) = 0 \rightarrow \quad 2 = 0$. This is an untrue statement. Therefore, there are no infection points. $f''(x) = 2$ is constant and positive. Therefore $f$ is concave up $(-\infty, \infty)$.

d) To sketch:



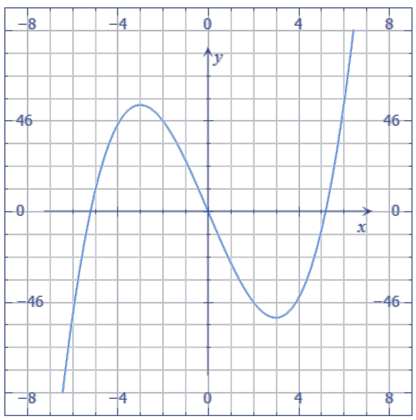
13) $f(x) = x^3 - 27x$

a) Let us first find all critical points of $f$: $f'(x) = 3x^2 - 27 = 0 \rightarrow \quad x = \pm 3$. So $(-3,0), (3,0)$ are our only critical points. $f'(-4) = 21$, $f'(0) = -27$, $f'(4) = 21$. Therefore, $f$ is increasing from $(-\infty, -3) \cup (3, \infty)$ and decreasing from $(-3,3)$.

b) $f$ has a local maximum at $x = -3$ and a local minimum at $x = 3$, from information in part a). (Note: see previous problem for explanation of why).

c) To find inflection point(s): $f''(x) = 0 \rightarrow \quad 6x = 0 \rightarrow \quad x = 0$. Therefore $(0,0)$ is our only candidate for inflection point. $f''(-1) = -6$ and $f''(1) = 6$. Therefore $f$ is concave down $(-\infty, 0)$, and concave up $(0, \infty)$.

d) To sketch: $x^3 - 27x$



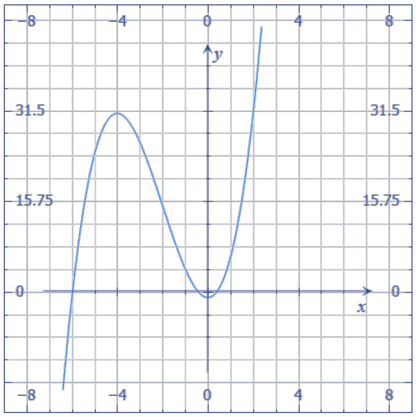

15) $f(x) = x^3 + 6x^2 - 1$

a) Let us first find all critical points of $f$: $f'(x) = 3x^2 + 12x = 0 \rightarrow \quad x = 0, -4$. So $(-4, -41), (0, -1)$ are our only critical points. $f'(-5) = 15$, $f'(-1) = -9$, $f'(1) = 15$. Therefore, $f$ is increasing from $(-\infty, -4) \cup (0, \infty)$ and decreasing from $(-4,0)$.

b) $f$ has a local maximum at $x = -4$ and a local minimum at $x = 0$, from information in part a).

c) To find inflection point(s): $f''(x) = 0 \rightarrow \quad 6x + 12 = 0 \rightarrow \quad x = -2$. Therefore $(-2,15)$ is our only candidate for an inflection point. $f''(-3) = -6$ and $f''(0) = 12$. Therefore $f$ is concave down $(-\infty, -2)$, and concave up $(-2, \infty)$.

d) To sketch:



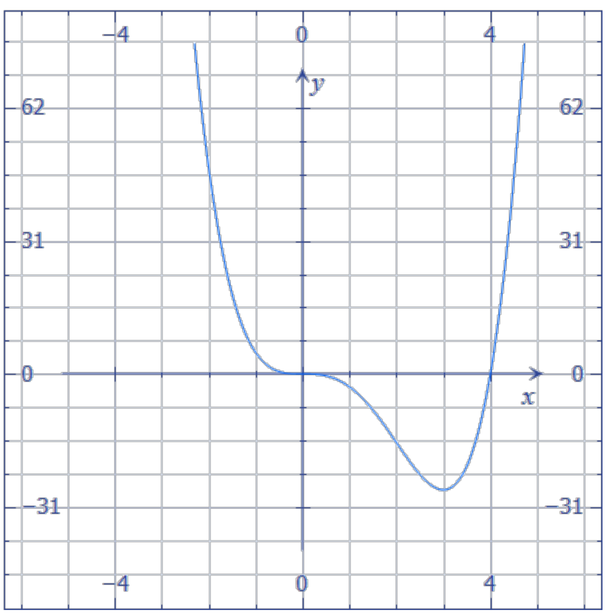

17) $f(x) = x^4 - 4x^3$

a) Let us first find all critical points of $f$: $f'(x) = 4x^3 - 12x^2 = 0 \rightarrow \quad x = 0{,}3$. So $(0{,}0), (3, -27)$ are our only critical points. $f'(-1) = -16$, $f'(1) = -8$, $f'(4) = 64$. Therefore, $f$ is decreasing from $(-\infty, 0) \cup (0{,}4)$ and increasing from $(3, \infty)$.

b) $f$ has a local minimum at $x = 3$, and a flat spot at $x = 0$ from information in part a).

c) To find inflection point(s): $f''(x) = 0 \rightarrow \quad 12x^2 - 24x = 0 \rightarrow \quad x = 0{,}2$. Therefore $(0{,}0), (2, -16)$ are our canditates for inflection points. $f''(-1) = 36$, $f''(1) = -12$ and $f''(3) = -180$. Therefore $f$ is concave down $(0{,}2)$, and concave up $(-\infty, 0) \cup (2, \infty)$.

d) To sketch:



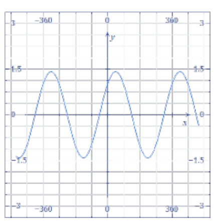
19) $f(x) = \sin x + \cos x$

a) Let us first find all critical points of $f$: $f'(x) = \cos x - \sin x = 0 \rightarrow \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$. So $\left(\frac{\pi}{4}, \sqrt{2}\right), \left(\frac{5\pi}{4}, -\sqrt{2}\right)$ are our only critical points. $f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}-1}{2}$, $f'\left(\frac{\pi}{2}\right) = -1$, $f'\left(\frac{3\pi}{2}\right) = 1$. Therefore, $f$ is increasing from $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$ and increasing from $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

b) $f$ has a local maximum at $x = \frac{\pi}{4}$, and a local minimum at $x = \frac{5\pi}{4}$ from information in part a).

c) To find inflection point(s): $f''(x) = 0 \rightarrow \; -\sin x - \cos x = 0 \rightarrow \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$. Therefore $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$ are candidates for our inflection points. $f''\left(\frac{\pi}{2}\right) = -1$, $f''(\pi) = 1$ and $f''\left(\frac{5\pi}{3}\right) = \frac{-\sqrt{3}-1}{2}$. Therefore $f$ is concave down $\left(0, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$ and concave up $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$.

d) To sketch:



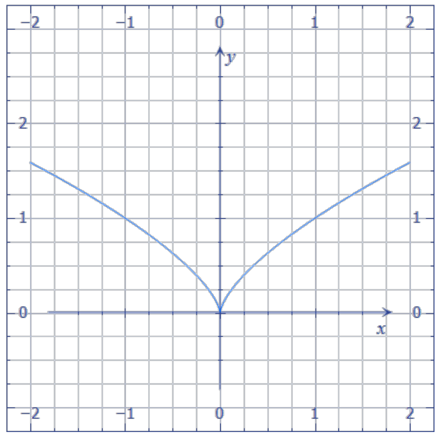

21) $f(x) = x^{\frac{2}{3}}$

a) Let us first find all critical points of $f$: $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = 0 \rightarrow \quad \frac{2}{3\sqrt[3]{x}} = 0 \rightarrow \quad 2 = 0$. So no critical points where the tangent line is horizontal. However, $x = 0$ is a value that $f'(x)$ does not exist, yet is in the domain of $f$. So, $(0,0)$ is our only critical point. $f'(-1) = 1$, $f'(1) = 1$, $f'(1) = 15$. Therefore, $f$ is decreasing from $(-\infty, 0)$ and increasing from $(0, \infty)$.

b) $f$ has a local minimum at $x = 0$, from information in part a).

c) To find inflection point(s): $f''(x) = 0 \rightarrow \; -\frac{2}{9}x^{-\frac{4}{3}} = 0 \rightarrow \quad -2 = 0$. So no solutions. However, at $x = 0$, $f''(x)$ does not exist, and is in the domain of $f$. So this is a candidate for an inflection point. $f''(-1) = -\frac{2}{9}$ and $f''(1) = -\frac{2}{9}$. Therefore $f$ is concave down $(-\infty, 0)$, and concave up $(0, \infty)$. (No inflection points).

d) To sketch:



−2
−1
0
1
2
y
x

# CHAPTER 3
## SECTION 4
## SUMMARY OF CURVE SKETCHING

**EXERCISES:**

Use the following steps to sketch the graph of the function:

a) $x$ and $y$ intercepts ($x$-intercepts for all, except polynomials over degree 2).
b) Vertical and horizontal asymptotes
c) 1st and 2nd derivatives
d) Critical points
e) Intervals of Increase or decrease
f) Local/Relative extrema
g) Inflection points
h) Intervals of concavity
i) Sketch

**(NOTE: A GRAPH WILL BE SHOWN FOR ALL ODD PROBLEMS. STEPS a)-i) WILL BE SHOWN FOR SELECT PROBLMES)**

1) $f(x) = 2x^2 + 4x + 2$:
   a) $x$ and $y$ intercepts: To find the $x$-intercept(s): $2x^2 + 4x + 2 = 0 \rightarrow \quad x^2 + 2x + 1 = 0 \rightarrow \quad x = -1$. To find the $y$-intercept: $2 \cdot 0^2 + 4 \cdot 0 + 2 = 2 \rightarrow \quad y = 2$.

   b) Vertical and horizontal asymptotes: $f$ is a polynomial, and has no vertical asymptotes (it is continuous everywhere). It also has no horizontal asymptotes (it has no denominator, or more precisely, the denominator is 1).

   c) 1st and 2nd derivatives: $f'(x) = 4x + 4, \quad f''(x) = 4$

   d) Critical points: We set $f'(x) = 0 \rightarrow \quad 4x + 4 = 0 \rightarrow \quad x = -1$, gives $(-1,0)$. Since $f$ is a polynomial, it is differentiable everywhere, so there is no value such that $f'(x)$ does not exist.

   e) Intervals of increase or decrease: We choose 3 values: Something less than $-1$, and something greater than $-1$: $f'(-2) = -4$, so $f$ is decreasing from $(-\infty, -1)$. $f'(0) = 4$, so $f$ is increasing from $(-1, \infty)$. We summarize :

   $f$ is decreasing from $(-\infty, -1)$.
   $f$ is increasing from $(-1, \infty)$.

f) Local Extrema: Our only critical point is: $(-1,0)$. $f$ was decreasing before $x = -1$, and increasing after. Therefore, $(-1,0)$ is a local minimum.

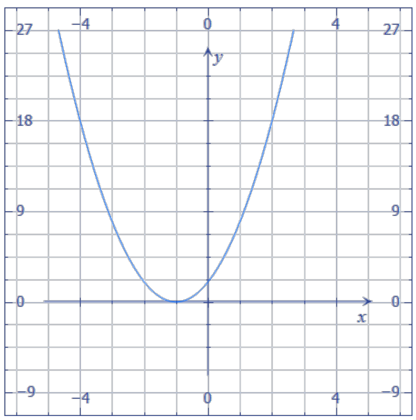
g) Inflection points: $f''(x) = 0 \rightarrow \quad 4 = 0$. So no inflection point. (Note: There are no values for which $f''(x)$ does not exist). Since $f''(x) = 4$, it is concave up everwhere.

h) Intervals of concavity: Now, we generalize what we found in part g):

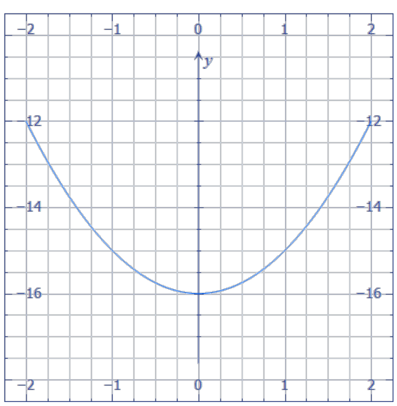
$f$ is concave up from $(-\infty, \infty)$ (since $f''(x) = 4$.

(We also note here that since $f''(x)$ is concave up everywhere, $(-1,0)$ is a local minimum. It is another way to find part f)).

i) Sketch:




3) $f(x) = x^2 - 16$:




5) $f(x) = x^3 + 9x^2$:

a) $x$ and $y$ intercepts: To find the $x$-intercept(s): $x^3 + 9x^2 = 0 \rightarrow \quad x = 0, -9$. To find the $y$-intercept: $0^3 + 9 \cdot 0^2 = 0 \rightarrow \quad y = 0$.

b) Vertical and horizontal asymptotes: $f$ is a polynomial, and has no vertical asymptotes (it is continuous everywhere). It also has no horizontal asymptotes (it has no denominator, or more precisely, the denominator is 1).

c) 1st and 2nd derivatives: $f'(x) = 3x^2 + 18x, \;\; f''(x) = 6x + 18$

d) Critical points: We set $f'(x) = 0 \rightarrow \quad 3x^2 + 18x = 0 \rightarrow \quad x = 0, -6$, gives $(0,0), (-6,108)$. Since $f$ is a polynomial, it is differentiable everywhere, so there is no value such that $f'(x)$ does not exist.

e) Intervals of increase or decrease: We choose 3 values: Something less than $-6$, something in the interval $(-6,0)$, and something greater than 0: $f'(-7) = 21$, so $f$ is increasing from $(-\infty, -6)$. $f'(-1) = -15$, so $f$ is decreasing from $(-6,0)$. $f'(1) = 21$, so $f$ is increasing from $(0, \infty)$. We summarize :

$f$ is increasing from $(-\infty, -6) \cup (0, \infty)$.
$f$ is decreasing from $(-6,0)$.

f) Local Extrema: Our only critical points are: $(-6,108), (0,0)$. $f$ was increasing before $x = -6$, and decreasing after. Therefore, $(-6,108)$ is a local maximum. $f$ was decreasing before $x = 0$, and increasing after. Therefore $(0,0)$ is a local minimum.

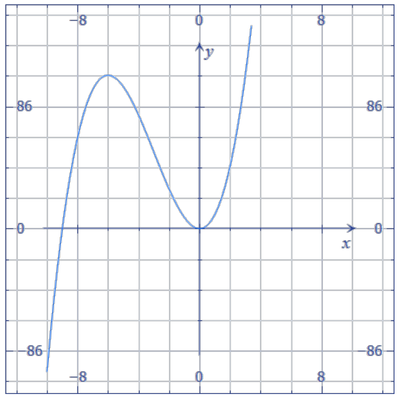
g) Inflection points: $f''(x) = 0 \rightarrow \quad 6x + 18 = 0 \rightarrow \quad x = -3, \;\; (-3,54)$. We check to see if $f$ changes concavity there: $f''(-4) = -6, \;\; f''(0) = 18$. $f$ changes concavity at $(-3,54)$. Therefore $(-3,54)$ is our only inflection point. (Note: There are no values for which $f''(x)$ does not exist).

h) Intervals of concavity: Now, we generalize what we found in part g):

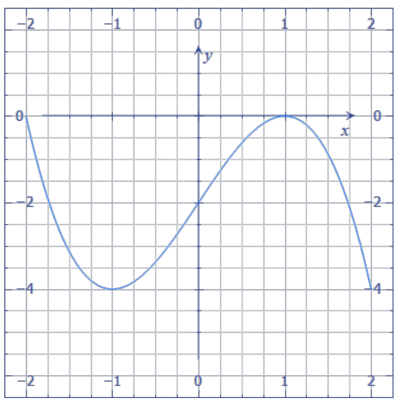
$f$ is concave down from $(-\infty, -3)$ (since $f''(-4) = -6$)
$f$ is concave up from $(-3, \infty)$ (since $f''(0) = 18$)

(We also note here that since $f''(-6)$ is concave down, it shows $(-6,108)$ is a local maximum, and since $f''(0)$ is concave up, it shows $(0,0)$ is a local minimum. It is another way to find part f)).

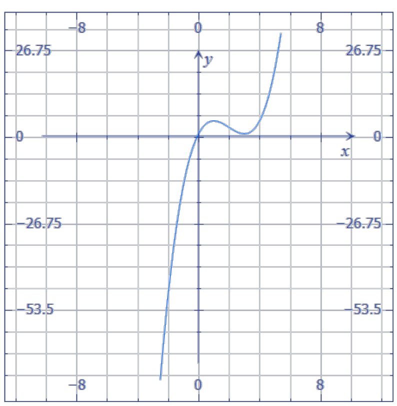
i) Sketch:



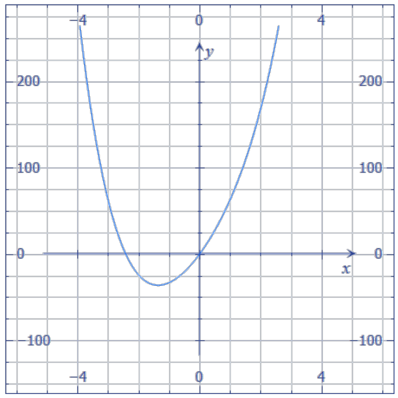

7) $f(x) = -x^3 + 3x - 2$:




9) $f(x) = x^3 - 6x^2 + 9x + 1$:




11) $f(x) = x^4 + 14x^2 + 48x$:




13) $f(x) = \cos x - \sin x,\ [0,2\pi]$:

a) $x$ and $y$ intercepts: $x$-intercept(s): $\cos x - \sin x = 0 \to \quad \cos x = \sin x \to \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$. To find the $y$-intercept: $\cos 0 - \sin 0 = 1$. So $y = 1$.

b) Vertical and horizontal asymptotes: There are no vertical or horizontal asymptotes:

c) 1st and 2nd derivatives: $f'(x) = -\sin x - \cos x,\ \ f''(x) = -\cos x + \sin x$

d) Critical points: We set $f'(x) = 0 \to \quad -\sin x - \cos x = 0 \to \quad -\sin x = \cos x \to \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$ gives $\left(\frac{3\pi}{4}, -\sqrt{2}\right),\ \left(\frac{7\pi}{4}, \sqrt{2}\right)$.

e) Intervals of increase or decrease: We choose 3 values: Something less than $\frac{3\pi}{4}$, something in the interval $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$, and something greater than $\frac{7\pi}{4}$: $f'\left(\frac{\pi}{2}\right) = -1$ is negative. Therefore $f$ is decreasing from $\left(0, \frac{3\pi}{4}\right)$. $f'(\pi) = 1$, so $f$ is increasing from $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$. $f'\left(\frac{11\pi}{6}\right) = \frac{1-\sqrt{3}}{2}$, so $f$ is decreasing from $\left(\frac{7\pi}{4}, 2\pi\right)$. We summarize:

$f$ is decreasing from $\left(0, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$
$f$ is increasing from $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$.

f) Local Extrema: Our only critical points are: $\left(\frac{3\pi}{4}, -\sqrt{2}\right),\ \left(\frac{7\pi}{4}, \sqrt{2}\right)$. $f$ is decreasing before $\frac{3\pi}{4}$ and increasing after, so $\left(\frac{3\pi}{4}, -\sqrt{2}\right)$ is a local minimum. $f$ is increasing before $\frac{7\pi}{4}$, and decreasing after, so $\left(\frac{7\pi}{4}, \sqrt{2}\right)$ is a local maximum.

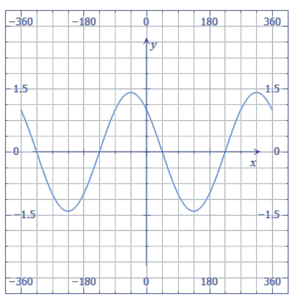
g) Inflection points: $f''(x) = 0 \to \quad -\cos x + \sin x = 0 \to \quad \sin x = \cos x \to \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$. So the two candidates for inflection points are $\left(\frac{\pi}{4}, 0\right),\ \left(\frac{5\pi}{4}, 0\right)$. Next, checking for changes in concavity at each point: $f''\left(\frac{\pi}{6}\right) = \frac{-\sqrt{3}+1}{2}$, so $f$ is concave down here. $f''\left(\frac{\pi}{2}\right) = 1$, so $f$ is concave up here. $f''\left(\frac{3\pi}{2}\right) = -1$, so $f$ is concave down here. So $f$ changes concavity at both $\left(\frac{\pi}{4}, 0\right)$, $\left(\frac{5\pi}{4}, 0\right)$, and these are our two inflection points.

h) Intervals of concavity: Now, we generalize what we found in part g):

$f$ is concave down from $\left(-\infty, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \infty\right)$
$f$ is concave up from $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

(We also note here that since $f''\left(\frac{3\pi}{4}\right)$ is concave up, it shows $\left(\frac{3\pi}{4}, -\sqrt{2}\right)$is a local minimum, and since $f''\left(\frac{7\pi}{4}\right)$ is concave down, it shows $\left(\frac{7\pi}{4}, \sqrt{2}\right)$ is a local maximum. It is another way to find part f)).

i) Sketch:




15) $f(t) = \sin^2 t,\ [0, 2\pi]$:

a) $x$ and $y$ intercepts: $x$-intercept(s): $\sin^2 t = 0 \rightarrow \quad \sin t = 0 \rightarrow \quad t = 0, \pi, 2\pi$. To find the $y$-intercept: $\sin^2 0 = 0$. So $y = 0$.

b) Vertical and horizontal asymptotes: There are no vertical or horizontal asymptotes:

c) 1st and 2nd derivatives: $f'(t) = 2\sin t \cos t, \quad f''(t) = -2\sin^2 t + 2\cos^2 t$

d) Critical points: We set $f'(t) = 0 \rightarrow \quad 2\sin t \cos t = 0 \rightarrow \quad t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. $0, 2\pi$ are not critical numbers because they are endpoints. This gives $\left(\frac{\pi}{2}, 1\right)$, $(\pi, 0)$, $\left(\frac{3\pi}{2}, 1\right)$ are our critical points.

e) Intervals of increase or decrease: We choose 4 values: Something less than $\frac{\pi}{2}$, something in the interval $\left(\frac{\pi}{2}, \pi\right)$, something in the interval $\left(\pi, \frac{3\pi}{2}\right)$and something greater than $\frac{3\pi}{2}$: $f'\left(\frac{\pi}{4}\right) = 1$ is positive. Therefore $f$ is increasing from $\left(0, \frac{\pi}{2}\right)$. $f'\left(\frac{3\pi}{4}\right) = -1$, so $f$ is decreasing from $\left(\frac{\pi}{2}, \pi\right)$. $f'\left(\frac{5\pi}{4}\right) = 1$, so $f$ is increasing from $\left(\pi, \frac{3\pi}{2}\right)$. $f'\left(\frac{7\pi}{4}\right) = -1$ so $f$ is decreasing from $\left(\frac{3\pi}{2}, 2\pi\right)$. We summarize:

$f$ is increasing from $\left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$
$f$ is decreasing from $\left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$.

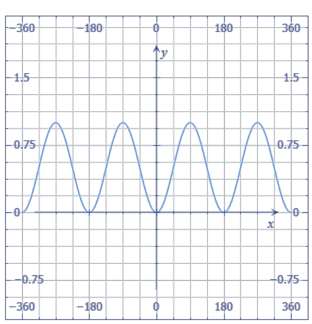
f) Local Extrema: Our only critical points are: $\left(\frac{\pi}{2}, 1\right)$, $(\pi, 0)$, $\left(\frac{3\pi}{2}, 1\right)$. $f$ is increasing before $\frac{\pi}{2}$ and decreasing after, so $\left(\frac{\pi}{2}, 1\right)$ is a local maximum. $f$ is decreasing before $\pi$, and increasing after, so $(\pi, 0)$ is a local minimum, $f$ is increasing before $\frac{3\pi}{2}$, and decreasing after, so $\left(\frac{3\pi}{2}, 1\right)$ is a local maximum.

g) Inflection points: $f''(x) = 0 \rightarrow \quad -2\sin^2 t + 2\cos^2 t = 0 \rightarrow \quad \sin^2 t = \cos^2 t \rightarrow \quad t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ So the two candidates for inflection points are $\left(\frac{\pi}{4}, \frac{1}{2}\right)$, $\left(\frac{3\pi}{4}\right)$, $\left(\frac{5\pi}{4}, \frac{1}{2}\right)$, $\left(\frac{7\pi}{4}, \frac{1}{2}\right)$. Next, checking for changes in concavity at each point: $f''(0) = 2$, so $f$ is concave up here. $f''\left(\frac{\pi}{2}\right) = -1$, so $f$ is concave down here. $f''(\pi) = 2$, so $f$ is concave up here. . $f''\left(\frac{3\pi}{2}\right) = -2$, so $f$ is concave down here. $f''(2\pi) = 2$, so $f$ is concave up here. So $f$ changes concavity at $\left(\frac{\pi}{4}, \frac{1}{2}\right)$, $\left(\frac{3\pi}{4}\right)$, $\left(\frac{5\pi}{4}, \frac{1}{2}\right)$, $\left(\frac{7\pi}{4}, \frac{1}{2}\right)$, and these are our four inflection points.

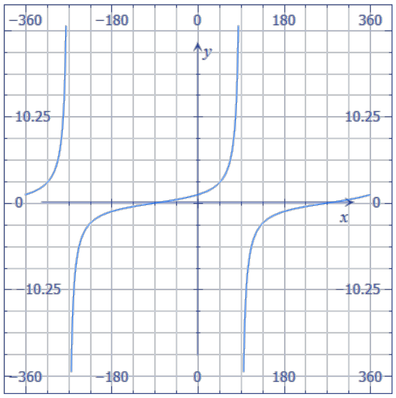
h) Intervals of concavity: Now, we generalize what we found in part g):

$f$ is concave up from $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$
$f$ is concave up from $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$.

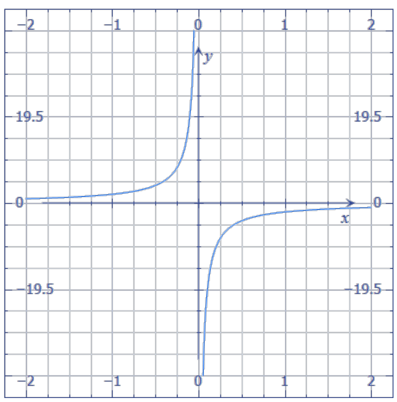
i) Sketch:



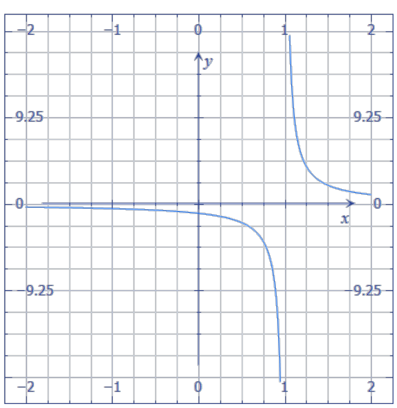
17) $f(x) = \frac{\cos x}{1-\sin x}$:




19) $f(x) = -\frac{2}{x}$:




21) $f(x) = \frac{2}{2x-2}$:



23) $f(x) = -\frac{5}{x+3}$:

a) $x$ and $y$ intercepts: $x$-intercept(s): $-\frac{5}{x+3} = 0 \rightarrow \; -1 = 0,$ so no $x$-intercepts. $y$-intercept: $y = -\frac{5}{3}$.

b) Vertical and horizontal asymptotes: Vertical asymptote: Set the denominator equal to zero: $x + 3 = 0 \rightarrow \quad x = -3.$ Since it does not cancel a factor in the numerator, it is our only vertical asymptote. Horizontal asymptote: $\lim_{x \to \pm\infty} -\frac{5}{x+3} = 0,$ so $y = 0$ is our horizontal asymptote.

c) 1st and 2nd derivatives: $f'(x) = \frac{1}{(x+3)^2}, \quad f''(x) = -\frac{2}{(x+3)^3}.$

d) Critical points: We set $f'(x) = 0 \rightarrow \; -\frac{1}{(x+3)^2} = 0 \rightarrow \; -1 = 0,$ so there are no places where $f$ has a horizontal tangent. There is a value where $f'(x)$ does not exist, but it is not in the domain of $f$, so no critical points.

e) Intervals of increase or decrease: The only place we have to check is on either side of the vertical asymptote, where $f$ is undefined: $f'(-4) = 1$ so $f$ is increasing here. $f'(0) = \frac{1}{9}$, so $f$ is also increasing here.

So $f$ is increasing from $(-\infty, -3) \cup (-3, \infty)$.

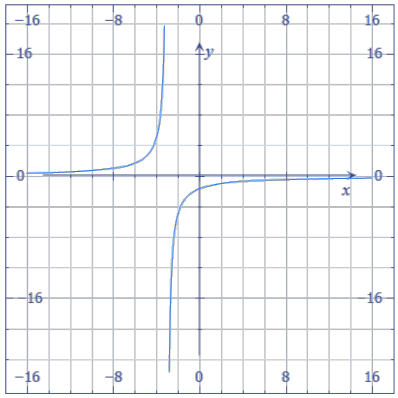
f) Local Extrema: There are no local extrema, since there are no critical points (recall local extrema can only occur at critical points).

g) Inflection points: $-\frac{2}{(x+3)^3} = 0 \rightarrow \; -2 = 0,$ so there are no inflection points such that $f''(x) = 0$. At $x = -3, \quad f''(x)$ does not exist, but it is also not in the domain of $f$. So there are no inflection points.

h) Intervals of concavity: There are no inflection points, but $f$ could change concavity at undefined values, so we check on either side of $x = -3$. $f''(-4) = 2$, so $f$ is concave up here. $f''(0) = -\frac{2}{27}$, so $f$ is concave down here.

$f$ is concave up from $(-\infty, -3)$
$f$ is concave down from $(-3, \infty)$.

i) Sketch:




25) $f(x)=\frac{1}{x^2-4}$:

a) $x$ and $y$ intercepts: $x$-intercept(s): $\frac{1}{x^2-4}=0 \rightarrow \quad 1=0$, so no $x$-intercepts. $y$-intercept: $y=\frac{1}{-4}=-\frac{1}{4}$.

b) Vertical and horizontal asymptotes: Vertical asymptote: Set the denominator equal to zero: $x^2-4=0 \rightarrow \quad x=\pm 2$. Since it does not cancel a factor in the numerator, our vertical asymptotes are $x=\pm 2$. . Horizontal asymptote: $\lim_{x\to\pm\infty}\frac{1}{x^2-4}=0$, so $y=0$ is our horizontal asymptote.

c) 1st and 2nd derivatives: $f(x)=(x^2-4)^{-1} \rightarrow$

$$f'(x)=-(x^2-4)^{-2}\cdot 2x=\frac{-2x}{(x^2-4)^2}.$$

$$f''(x)=\frac{(x^2-4)^2(-2)-2(x^2-4)\cdot(2x)\cdot(-2x)}{(x^2-4)^4}=\frac{-2(x^2-4)+8x^2}{(x^2-4)^3}=\frac{6x^2+2}{(x^2-4)^3}$$

d) Critical points: We set $f'(x)=0 \rightarrow \quad \frac{-2x}{(x^2-4)^2}=0 \rightarrow \quad 2x=0 \rightarrow \quad x=0$. So we have a critical point at $(0,-1)$. At $x=\pm 2$, $f'(x)$ does not exist, but neither are in the domain of $f$. So $\left(0,-\frac{1}{4}\right)$ is our only critical point.

e) Intervals of increase or decrease: We need to check on either side of our critical point, as well as on either side of both undefined values (in this case, our vertical asymptotes). $f'(-3)=$ positive, so $f$ is increasing here. $f'(-1)=$ also positive. $f'(1)=$ negative, so $f$ is decreasing here. $f'(3)=$ also negative.

$f$ is increasing from $(-\infty,-2)\cup(-2,0)$
$f$ is decreasing from $(0,2)\cup(2,\infty)$.

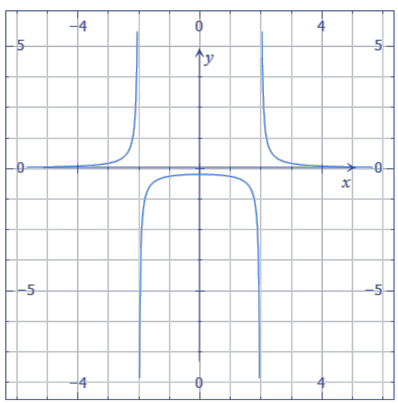
f) Local Extrema: $f'(x)$ is increasing before $x = 0$, and decreasing after, so $\left(0, -\frac{1}{4}\right)$ is a local maximum.

g) Inflection points: $f''(x) = 0 \rightarrow \frac{6x^2+2}{(x^2-4)^3} \rightarrow 6x^2 = -2$ means no inflection points. $f''(x)$ does not exist at $x = \pm 2$, but they are not in the domain of $f$.

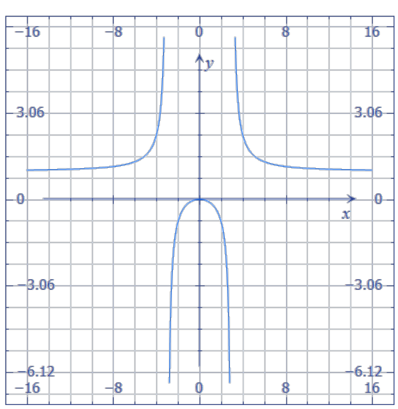
h) Intervals of concavity: We have no inflection points, but we need to check for changes in concavity around the undefined values (vertical asymptotes): $f''(-3) =$ positive, so $f$ is concave up, $f''(0) =$ negative, so $f$ is concave down, and $f''(3) =$ positive, and $f$ is concave up again.

$f$ is concave up $(-\infty, -2) \cup (2, \infty)$
$f$ is concave down $(-2,2)$.

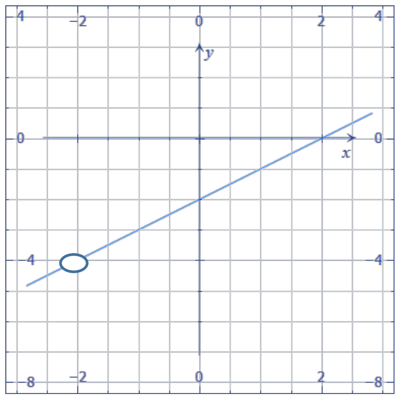
i) Sketch:




27) $f(x) = \frac{x^2}{x^2-9}$:




29) $f(x) = \frac{x^2-4}{x+2}$:




31) $f(x) = 2(x-2)^{\frac{1}{3}}$:

a) $x$ and $y$ intercepts: $x$-intercept(s): $\left(0, -2^{\frac{4}{3}}\right)$ is the $y$-intercept. $2(x-2)^{\frac{1}{3}} = 0 \rightarrow \quad x - 2 = 0 \rightarrow \quad x = 2$ is the $x$-intercept.

b) Vertical and horizontal asymptotes: There are no vertical or horizontal asymptotes.

c) 1st and 2nd derivatives: $f'(x) = \frac{1}{3}(x-2)^{-\frac{2}{3}} = \frac{1}{3(x-2)^{\frac{2}{3}}}, \quad f''(x) = -\frac{2}{9}(x-2)^{-\frac{5}{3}} = -\frac{2}{9(x-2)^{\frac{5}{3}}}$.

a. Critical points: We set $f'(x) = 0 \rightarrow \frac{1}{3(x-2)^{\frac{2}{3}}} = 0 \rightarrow \quad 1 = 0$ so there are no critical points for which $f$ has a horizontal tangent line. However, when $x = 2$, it is a value for which $f'(x)$ does not exist, and this one is in the domain of $f$. Therefore, $(2,0)$ is a critical point of $f$.

b. Intervals of increase or decrease: We check on either side of our critical point $(2,0)$. $f'(0) = \text{positive}$, so $f$ is increasing, and $f'(3) = \text{positive}$, so $f$ is increasing.

$f$ is increasing from $(-\infty, 2)$
$f$ is increasing from $(2, \infty)$.

c. Local Extrema: $f'(x)$ is increasing before $x = 2$, and increasing after, so no local extrema.

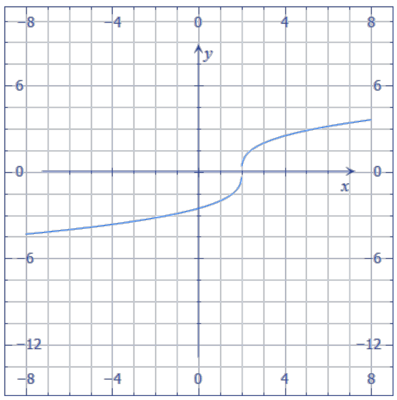
d. Inflection points: $f''(x) = 0 \rightarrow \quad -\frac{2}{9(x-2)^{\frac{5}{3}}} = 0 \rightarrow \quad -2 = 0$ which does not give an inflection point. However at $x = 2, \quad f''(x)$ does not exist, and is in the domain of $f$.

Therefore, we will check for concavity changes on either side of 2: We notice that $f''(0)$ is always positive, and $f''(3)$ is negative. Therefore $(-2,0)$ is our only inflection point.

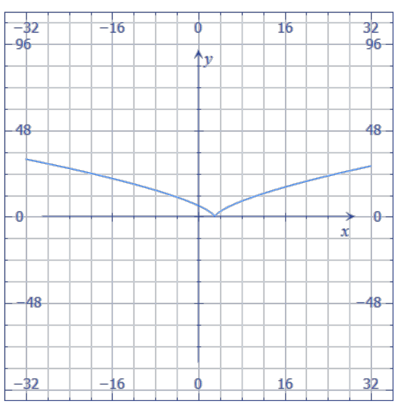
d) Intervals of concavity: We notice from part g), that:

$f$ is concave up from $(-\infty, 2)$
$f$ is concave down $(2, \infty)$.

e) Sketch:




33) $f(x) = 3(x-3)^{\frac{2}{3}}$:

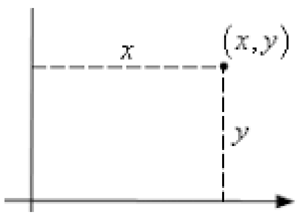


CHAPTER 3
SECTION 5
APPLICATIONS: OPTIMIZATION

**EXERCISES:**

1) Of all the rectangles whose perimeter is 50 cm, find the dimensions of the rectangle with the maximum area? What is the maximum area?

   a) First, we draw a picture.




   b) We have also labeled the picture.

   c) Here, we need two expressions:

      i) We need a function that we are trying to maximize: In this case, it is Area: $A = xy$.
      ii) In this case we have 2 variables, so we need an equation the relates them: $2x + 2y = 50$.

   d) Let us solve for $y$: $y = 25 - x$: (Note: It didn't matter which variable we chose to solve for, as either would lead to the same result).

   e) We substitute d) into our area function: $A = x(25 - x) = 25x - x^2$.

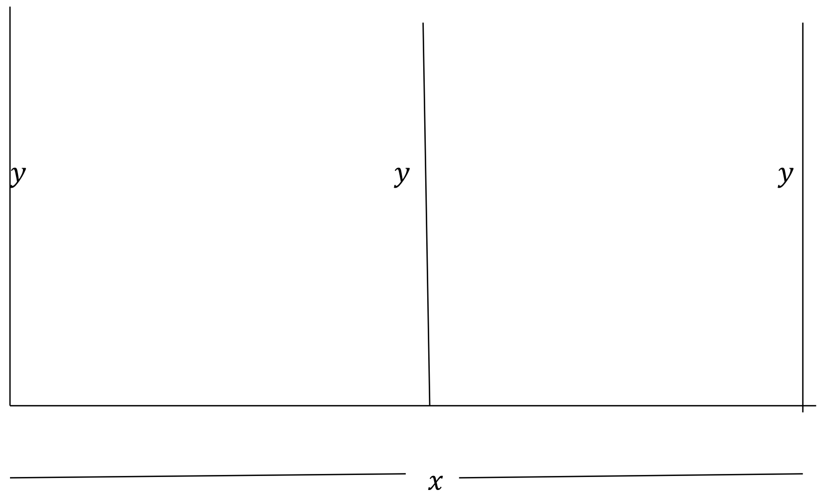
   f) We find all critical points: $A' = 25 - 2x = 0 \rightarrow \quad x = 12.5\,ft$. When $x = 12.5$,

   $y = 25 - 12.5 = 12.5$. The area is $12.5 \times 12.5 = 156.25\,ft^2$.

   g) We will use the second derivative to show that $x = 12.5$ is a maximum and not a minimum: $A'' = -2$, which is always concave down. Therefore, $(12.5, 12.5)$ is a local maximum.

3) Anna is building a fence for her two dogs. She wants to make two separate areas for them, with fencing down the middle. She will use her house as one side of the area. Joyce has 300 ft. of fencing. What are the dimensions that will maximize the area, and what is the total area that will be enclosed?

a) First, we draw a picture:




b) We have also labeled the picture. (Note: I have labeled the whole horizontal side, $x$, instead of $2x$. This way, we can avoid some fractions.)

c) Here, we need two expressions:

i) We need a function that we are trying to maximize: In this case, it is Area: $A = xy$.
ii) In this case we have 2 variables, so we need an equation the relates them: $x + 3y = 300$.

d) In this case, we choose to solve for $x$ to avoid fractions: $x = 300 - 3y$.

e) We substitute 4) into our area function: $A = (300 - 3y)y = 300y - 3y^2$.

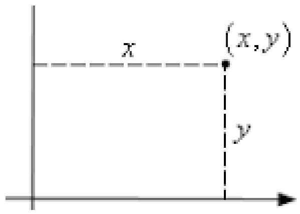
f) We find all critical points: $A' = 300 - 6y = 0 \rightarrow \quad y = 50\ ft$. When $y = 50$,

$x = 300 - 3 \cdot 50 = 150\ ft$. $A = 50 \times 150 = 7500\ ft^2$.

g) We will use the second derivative to show that $y = 50$ is a maximum and not a minimum: $A'' = -6$, which is always concave down. Therefore, $(150,50)$ is a local maximum.

5) Fred is building a rectangular fence for his garden. His garden must have an area of 12,000 $ft^2$ in order for his plan to have enough room. He wants to minimize the cost of the materials. What are the dimensions that will minimize the cost?

a) First, we draw a picture.




b) We have also labeled the picture.

c) Here, we need two expressions:

i) We need a function that we are trying to minimize: In this case, it is Perimeter: $2x + 2y$.
ii) In this case we have 2 variables, so we need an equation the relates them: $A = xy = 12{,}000$.

d) Let us solve for $y$: $y = \frac{12{,}000}{x}$: (Note: It didn't matter which variable we chose to solve for, as either would lead to the same result).

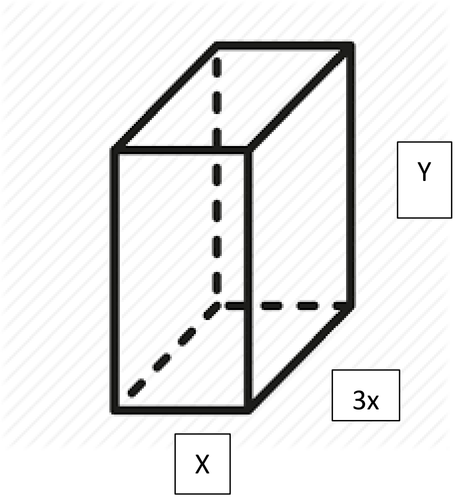
e) We substitute d) into our perimeter function: $P = 2x + 2 \cdot \frac{12{,}000}{x} = 2x + \frac{24{,}000}{x}$.

f) We find all critical points: $P' = 2 - \frac{24{,}000}{x^2} = 0 \rightarrow \quad x^2 = 12{,}000 \rightarrow \quad x \approx 109.5$. When $x = 109.5$, $y = \frac{12{,}000}{109.5} = 109.5$. So the perimeter that will minimize cost is $109.5 \times 109.5$.

g) We will use the second derivative to show that $x = 109.5$ is a minimum and not a maximum:
$P'' = \frac{48{,}000}{x^3}$, which is always concave up when $x$ is positive. Therefore, $(109.5, 109.5)$ is a local minimum.

7) Jane is building a rectangular container out of wood, including a top, to store some household items. Jane has a fixed surface area of $100\ ft^2$. She wants to maximize the volume. The length is three times the width. What are dimensions that will maximize the volume, and what is the volume?

a) First, we draw a picture:




b) We have also labeled the picture.
c) Here, we need two expressions:

i) We need a function that we are trying to maximize: In this case, it is Volume: $V = 3x^2y$.
ii) In this case we have 2 variables, so we need an equation the relates them:

$3x^2 + 3x^2 + 2xy + 3xy + 3xy = 6x^2 + 8xy = 100.$

d) In this case we solve for $y$: $y = \frac{100-6x^2}{8x}$.

e) We substitute d) into our volume function: $V = 3x^2\left(\frac{100-6x^2}{8x}\right) = \frac{3}{8}(300x - 18x^3)$.

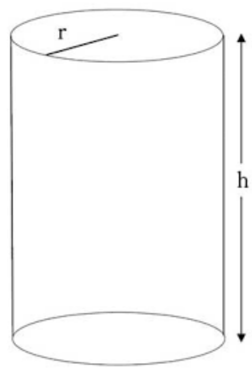
f) We find all critical points: $V' = \frac{3}{8}(300 - 54x^2) = \frac{225}{2} - \frac{81}{4}x^2 = 0 \rightarrow \quad x^2 = \frac{225}{2} \cdot \frac{4}{81} = \frac{50}{9} \rightarrow \quad x \approx \pm 2.36$.

$x$ cannot be $-2.36$, because length cannot be negative. So $x = 2.36\ ft$, $y \approx 3.53\ ft$. Volume is $2 \cdot 2.36^2 \cdot 3.53 \approx 39.3 ft^3$.

g) We will use the second derivative to show that $x = 2.36$ is a maximum and not a minimum: $V'' = -\frac{81}{2}4x \rightarrow \quad V''$ is negative, which is concave down. Therefore, $(2.36, 3.53)$ is a local maximum.

9) Mindy has a soup company. She is trying to save money on the materials she is using for her cans. She uses cylindrical cans to contain her soup. If her can has a volume of $570\ cm^3$, what dimensions will minimize the surface are, and what is the surface area?

a) We draw a picture:




b) We have also labeled the picture.

c) Here, we need two expressions:

i) We need a function that we are trying to minimize: In this case, it is Surface Area:
ii) $S = 2\pi r^2 + 2\pi rh$.
iii) In this case we have 2 variables, so we need an equation the relates them: $V = \pi r^2 h = 570$.

d) Here we solve for $h$: $h = \frac{570}{\pi r^2}$.

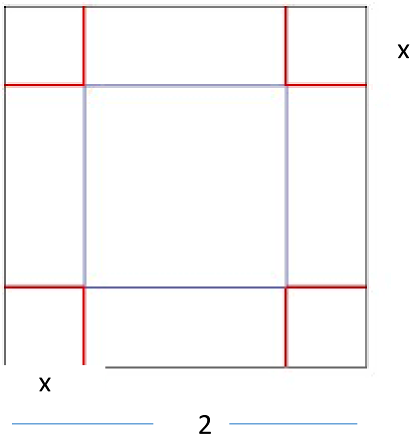
e) We substitute 4) into our surface area function: $S = 2\pi r^2 + 2\pi r \cdot \left(\frac{570}{\pi r^2}\right) = 2\pi r^2 + \frac{1140}{r}$.

f) We find all critical points: $S' = 4\pi r - \frac{1140}{r^2} = 0 \to \quad \frac{1140}{r^2} = 4\pi r \to \quad 4\pi r^3 = 1140 \to$
$r^3 = \frac{285}{\pi} \to \quad r \approx 4.49. \quad h \approx 9.$

g) $S'' = 4\pi + \frac{2280}{r^3}$ is always positive, since $r$ is always positive. This makes $S$ concave up whenever $r > 0$, so we have a local minimum.

11) Bob is making a box with rectangular box out of a squarer sheet of cardboard. The sides are $2 \times 2\ ft$. He is cutting out the corners, so he can fold it up and tape it together. What size cut should he make to maximize the volume of the box?

a) First we draw a picture;




b) We have already labeled the picture.

c) In this case we only need one expression for Volume: $V = (2-2x)(2-2x)x =$

$x(4-8x+4x^2) = 4x^3 - 8x^2 + 4x.$

d) We can skip this

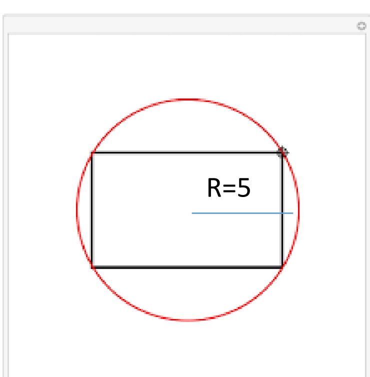
e) And this

f) We skip ahead to critical points: $V' = 12x^2 - 16x + 4 = 0 \rightarrow \quad 3x^2 - 4x + 1 = 0 \rightarrow$ $x = \frac{1}{3}, 1.$ We immediately see that we have to take $\frac{1}{3}$, as the other value = 2 when doubled, which gives a zero volume.

g) $V'' = 24x - 16.$ At $x = \frac{1}{3}$, $V'' = -8.$ So $V$ is concave down, and a maximum.

13) Enclose a rectangle within a circle. Find the maximum area of the rectangle that lies within a circle of radius 5.

a) First, we draw a picture.



b) We have also labeled the picture. (Also let $x =$ the horizontal side of the rectangle, and $y =$ the vertical side.

c) Here, we need two expressions:

iii) We need a function that we are trying to maximize: In this case, it is Area of the rectangle: $A = xy$.

iv) In this case we have 2 variables, so we need an equation the relates them: We use the equation of the circle: $x^2 + y^2 = 25$.

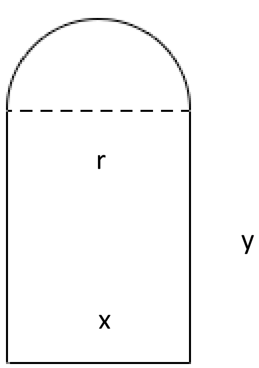
d) Let us solve for $y$: $y = \sqrt{25 - x^2}$: (Note: It didn't matter which variable we chose to solve for, as either would lead to the same result).

e) We substitute d) into our area function: $A = x\sqrt{25 - x^2} = \sqrt{25x^2 - x^4}$.

f) We find all critical points: $A' = \frac{1}{2}(25x^2 - x^4)^{-\frac{1}{2}}(50x - 4x^3) = \frac{50x - 4x^3}{2\sqrt{25x^2 - x^4}} = 0 \rightarrow \quad 50x - 4x^3 = 0 \rightarrow \quad 2x(25 - 2x^2) = 0 \rightarrow \quad x = 0, \pm\frac{5\sqrt{2}}{2}$. We will not choose $x = 0$ for obvious reasons, and the same for $-\frac{5\sqrt{2}}{2}$. Therefore, $x = \frac{5\sqrt{2}}{2}$, $y = \sqrt{25 - \left(\frac{5\sqrt{2}}{2}\right)^2} = \sqrt{25 - 25 \cdot \frac{2}{4}} = \sqrt{\frac{25}{2}} = \frac{5\sqrt{2}}{2}$.

g) The second derivative would show that $x = \frac{5\sqrt{2}}{2}$ is a maximum and not a minimum:

15) Sarah is building her first house. She wants to put in a Norman window. This window is a rectangular, with a semi-circle on top. The perimeter will be 50 ft. What are the dimensions that will maximize the area, and what is the maximum area:




a) First, we draw a picture.

b) We have also labeled the picture.

c) Here, we need two expressions:

i) We need a function that we are trying to maximize: In this case, it is Area: $xy + \pi r^2$. Also $r = \frac{x}{2}$ so $A = xy + \frac{\pi x^2}{4}$

ii) In this case we have 2 variables, so we need an equation the relates them: $P = x + 2y + \frac{2\pi x}{2} = x + 2y + \pi x = 50$

d) Let us solve for $y$: $y = \frac{50 - x - \pi x}{2}$:

e) We substitute d) into our area function: $A = x\left(\frac{50 - x - \pi x}{2}\right) + \frac{\pi x^2}{4} = 25x - \frac{x^2}{2} - \frac{\pi x^2}{2} + \frac{\pi x^2}{4} = 25x - \frac{x^2}{2} - \frac{\pi x^2}{4}$.

f) We find all critical points: $A' = 25 - x - \frac{\pi x}{2} = 0 \rightarrow \quad x\left(1 + \frac{\pi}{2}\right) = 25 \rightarrow \quad x = \frac{25}{\left(1 + \frac{\pi}{2}\right)} \approx 9.72$. When $x = 9.72$, $y \approx 4.87$. $r \approx 4.86$. And, maximum area is $9.72 \cdot 4.87 + \pi \cdot 4.86^2 \approx 121.5\ ft^2$

g) We will use the second derivative to show that $x = 9.72$ is a maximum and not a minimum:

$A'' = -1 - \frac{\pi}{2}$, which is always concave down. Therefore, $(9.72, 4.86)$ is a local maximum.

17) Find the point on the graph of $y = 3x - 2$ that is closest to $(0,0)$.

For this problem, we need the distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This involves a square root. We can minimize the distance squared instead of the distance to obtain the same result.

a) We do not need a picture for this.

b) Our variables are already chosen.

c) We want to minimize $d^2 = y^2 + x^2$.

d) This step is already done.

e) We substitute $y = 3x - 2$ for $y$: $(3x - 2)^2 + x^2 = 9x^2 - 12x + 4 + x^2 = 10x^2 - 12x + 4$.

f) We find all critical points: $(d^2)' = 20x - 12 = 0 \rightarrow \quad x = \frac{12}{20} = \frac{3}{5}$

So $\left(\frac{3}{5}, -\frac{1}{5}\right)$ is our only critical point.

g) $(d^2)'' = 20$ is concave up always. So, $\left(\frac{3}{5}, -\frac{1}{5}\right)$ is a local minimum.

19) Find the point t on the graph of $y = \sqrt{x}$ that is closest to $(1,0)$.

For this problem, we need the distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This involves a square root. We can minimize the distance squared instead of the distance to obtain the same result.

a) We do not need a picture for this.

b) Our variables are already chosen.

c) We want to minimize $d^2 = y^2 + (x-1)^2$.

d) This step is already done.

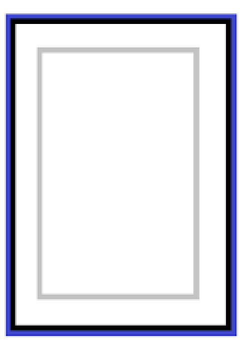
e) We substitute $y = \sqrt{x}$ for $y$: $\left(\sqrt{x}\right)^2 + (x-1)^2 = x + x^2 - 2x + 1 = x^2 - x + 1$.

f) We find all critical points: $(d^2)' = 2x - 1 = 0 \rightarrow \quad x = \frac{1}{2}$

So $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$ is our only critical point.

g) $(d^2)'' = 2$ is concave up always. So, $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$ is a local minimum.

21) A poster will have a total area of $1150\ cm^2$. If the margins are $3\ cm$ all around, what dimensions will give the largest printed area, and what is the printed area?



a) First, we draw a picture.

b) Let $x =$ the horizontal side of the printed area, and $y =$ the vertical side of the printed area.

c) Here, we need two expressions:

i) We need a function that we are trying to minimize: In this case, it is Area of the rectangle: $A = xy$.

ii) In this case we have 2 variables, so we need an equation the relates them: We use the area of the total poster: $(x+6)(y+6) = 1150$.

d) Let us solve for $y$: $y = \frac{1150}{x+6} - 6$: (Note: It didn't matter which variable we chose to solve for, as either would lead to the same result).

e) We substitute d) into our area function: $A = x\left(\frac{1150}{x+6} - 6\right) = \frac{1150x}{x+6} - 6x$.

f) We find all critical points: $A' = \frac{(x+6)\cdot 1150 - 1150x}{(x+6)^2} - 6 = 0 \rightarrow \frac{1150x + 6900 - 1150x}{(x+6)^2} - 6 = 0 \rightarrow$ $6900 = 6(x+6)^2 \rightarrow \quad 6900 = 6(x^2 + 12x + 36) \rightarrow \quad 1150 = x^2 + 12x + 36 \rightarrow$ $x^2 + 12x - 1114 \rightarrow \quad x = \frac{-12 \pm \sqrt{144 - 4(-1114)}}{2} = \frac{-12 \pm \sqrt{4600}}{2} \approx \frac{(-12 + 67.8)}{2} \approx 27.9$. We will not choose $x = \frac{-12 - \sqrt{4600}}{2}$ for obvious reasons. Therefore, $x = 27.9$, $y = 27.9$. So, the total area of the printed area will be $27.9 \times 27.9 = 778.4$

g) The second derivative would show that $x = 27.9$ is a maximum and not a minimum:

23) Hunter is opening a clothing line of jeans. The price, $p$ is modeled by $p = 150 - 0.5x$. The total cost of producing $x$ dresses is $C(x) = 1500 + 0.5x^2$.

a) Find the total revenue, $R(x)$. $R(x) = p \cdot x = 150x - 0.5x^2$

b) Find the total profit, $P(x)$. $P(x) = R(x) - C(x) = 150x - 0.5x^2 - (1500 + 0.5x^2) = -x^2 + 150x - 1500$

c) How many jeans must Hunter sell in order in order to maximize his profit? We find the derivative, and set it equal to zero: $P'(x) = -2x + 150 = 0 \rightarrow \quad x = 75$

d) What is the maximum profit? $P(75) = -75^2 + 150 \cdot 75 - 1500 = \$4125$

e) What price per stove must be changed to maximize profit? $p = 150 - 0.5 \cdot 75 = \$112.50$

# CHAPTER 3
# SECTION 6
# INDEPTERMINATE FORMS OF LIMITS USING L'HOSPITAL'S RULE
# NEWTON'S METHOD

**EXERCISES (For L'Hospital's Rule):**

Find the following limits. Use L'Hospital's Rule whenever applicable: (You can use an easier method if it saves time)

1) $\lim\limits_{x\to\infty} \frac{2x+9}{7x-7} = \lim\limits_{x\to\infty} \frac{2}{7} = \frac{2}{7}$

3) $\lim\limits_{x\to\infty} \frac{3x^2+2x}{x-8} = \lim\limits_{x\to\infty} \frac{6x+2}{1} = \infty$

5) $\lim\limits_{x\to\infty} \frac{4x-17}{3x^2-12x+2} = \lim\limits_{x\to\infty} \frac{4}{6x-12} = 0$

7) $\lim\limits_{x\to\infty} \frac{3x^2-9x+2}{x^3-7x^2+2x-4} = \lim\limits_{x\to\infty} \frac{6x-9}{3x^2-14x+2} = \lim\limits_{x\to\infty} \frac{6}{6x-14} = 0$

9) $\lim\limits_{x\to 2^+} \frac{x^3-19x}{x-2} = \lim\limits_{x\to 2^+} -\frac{30}{2^+-2} = -\infty$

11) $\lim\limits_{x\to\infty} \frac{\ln\sqrt{x}}{2x} = \lim\limits_{x\to\infty} \frac{1}{4\sqrt{x}} = 0$

13) $\lim\limits_{x\to\infty} \frac{x^2-19x}{e^{2x}} = \lim\limits_{x\to\infty} \frac{2x-19}{2e^{2x}} = \lim\limits_{x\to\infty} \frac{2}{4e^{2x}} = 0$

15) $\lim\limits_{x\to 0^-} \frac{e^x-2^x}{x^2} = \lim\limits_{x\to 0^-} \frac{e^x-2^x\ln 2}{2x} = -\infty$

17) $\lim\limits_{x\to\frac{\pi}{2}} \frac{2\sin x-2}{4\cos 4x} = \lim\limits_{x\to\frac{\pi}{2}} -\frac{2\cos x}{4\sin 4x} = 0$

19) $\lim\limits_{x\to\infty} \frac{\ln x^4}{x^2} = \lim\limits_{x\to\infty} \frac{4\ln x}{x^2} = \lim\limits_{x\to\infty} \frac{2}{x^2} = 0$

21) $\lim\limits_{x\to\infty} x\, e^{-2x} = \lim\limits_{x\to\infty} \frac{x}{e^{2x}} = \lim\limits_{x\to\infty} \frac{1}{2e^{2x}} = 0$

23) $\lim\limits_{x\to\infty} x\sin\frac{2}{x} = \lim\limits_{x\to\infty} \frac{\sin\frac{2}{x}}{\frac{1}{x}} = \lim\limits_{x\to\infty} \frac{2\cos\frac{2}{x}}{x^2\left(\frac{1}{x^2}\right)} = (\text{Note: there are two}-\text{'s}) = 2\lim\limits_{x\to\infty}\cos\frac{2}{x} = 2$

25) $\lim\limits_{x\to 0^+} 2x^3\ln x^2 = 4\lim\limits_{x\to 0^+} x^3\ln x = 4\lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{x^3}} = -4\lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{\frac{3}{x^4}} = 0$

27) $\lim_{x\to 0^+} \ln x \sin 2x = \lim_{x\to 0^+} \frac{\ln x}{(\sin 2x)^{-1}} = -\lim_{x\to 0^+} \frac{(\sin 2x)^2}{2x\cos 2x} = \frac{1}{2}\lim_{x\to 0^+} \frac{4\sin 2x \cos 2x}{4x\sin 2x + 2\cos 2x} = 0$

29) $\lim_{x\to 0^+} \frac{3}{x^2} - \frac{1}{x+3} = \lim_{x\to 0^+} \frac{3x+9-x^2}{x^2(x+3)} = \infty$

31) $\lim_{x\to 0^+} \frac{1}{x} - \frac{2}{2x^2+x} = \lim_{x\to 0^+} \frac{2x^2+x-2x}{x(2x^2+x)} = \lim_{x\to 0^+} \frac{2x^2-x}{2x^3+x^2} = \lim_{x\to 0^+} \frac{\frac{2}{x}-\frac{1}{x^2}}{2+\frac{1}{x}} = 0$

33) $\lim_{x\to 1^+} \frac{3}{x^2-1} - \frac{2}{\ln x} = \lim_{x\to 1^+} \frac{3\ln x - 2(x^2-1)}{\ln x\,(x^2-1)} = \lim_{x\to 1^+} \frac{\frac{3}{x}-4x}{x^2\ln x + x - \frac{1}{x}} = -\infty$

35) $\lim_{x\to 0^+} x^{2x}$:

We let $y = \lim_{x\to 0^+} x^{2x}$. We take $ln$ of both sides to get: $\ln y = \ln \lim_{x\to 0^+} x^{2x} = \lim_{x\to 0^+} \ln x^{2x}$, by limit law. Then, $\lim_{x\to 0^+} \ln x^{2x} = 2\lim_{x\to 0^+} x\ln x$, by a rule of logarithms. We now have an indeterminate product, which we rewrite as $2\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$, which is now in a form of $-\frac{\infty}{\infty}$. We can now apply L'Hospital's Rule to get: $2\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -2\lim_{x\to 0^+} \frac{x^2}{x} = -2\lim_{x\to 0^+} x = 0$. So we calculated $\ln \lim_{x\to 0^+} x^{2x}$. But we wanted $\lim_{x\to 0^+} x^{2x}$. We have $\ln y = \ln \lim_{x\to 0^+} x^{2x} = 0$. Therefore $y = \lim_{x\to 0^+} x^{2x} = e^0 = 1$.

37) $\lim_{x\to 0^+} (1+\sin x)^{\frac{1}{x}}$:

We let $y = \lim_{x\to 0^+} (1+\sin x)^{\frac{1}{x}}$. We take $ln$ of both sides to get: $\ln y = \ln \lim_{x\to 0^+} (1+\sin x)^{\frac{1}{x}} = \lim_{x\to 0^+} \ln(1+\sin x)^{\frac{1}{x}}$, by limit law. Then, $\lim_{x\to 0^+} \ln(1+\sin x)^{\frac{1}{x}} = \lim_{x\to 0^+} \left(\frac{1}{x}\right)\ln(1+\sin x)$, by a rule of logarithms. We now have a form of $\frac{0}{0}$. We can now apply L'Hospital's Rule to get: $\lim_{x\to 0^+} \frac{\cos x}{1+\sin x} = 1$. So we calculated $\ln \lim_{x\to 0^+} (1+\sin x)^{\frac{1}{x}}$. But we wanted $\lim_{x\to 0^+} (1+\sin x)^{\frac{1}{x}}$. We have $\ln y = \ln \lim_{x\to 0^+} (1+\sin x)^{\frac{1}{x}} = 1$. Therefore $y = \lim_{x\to 0^+} (1+\sin x)^{\frac{1}{x}} = e^1 = e$.

39) $\lim_{x\to\infty} 2x^{e^{-2x}} = 2\lim_{x\to\infty} \frac{x}{e^{2x}} = 2\lim_{x\to\infty} \frac{1}{2e^{2x}} = 0$

**CHAPTER 3**
**SECTION 7**
**ANTIDERIVATIVES:**

**EXERCISES:**

Find the general antiderivative for the following functions:

1) $f(x) = x - 9$: $F(x) = \frac{x^2}{2} - 9x + C$

3) $f(x) = 2x^2 - 5x$: $F(x) = \frac{2x^3}{3} - \frac{5x^2}{2} + C$

5) $f(x) = x^4 + 3x^3 + \frac{1}{2}x^2$: $F(x) = \frac{x^5}{5} + \frac{3x^4}{4} + \frac{x^3}{6} + C$

7) $f(x) = \sec^2 x - 3e^x + 7$: $F(x) = \tan x - 3e^x + 7x + C$

9) $f(x) = \sec x \tan x + \frac{1}{2}e^x$: $F(x) = \sec x + \frac{1}{2}e^x + C$

11) $f(x) = \frac{4x^2 - x^{-1} + x}{x} = 4x - x^{-2} + 1$: $F(x) = 2x^2 - \frac{x^{-3}}{-3} + x + C = 2x^2 + \frac{1}{3x^3} + x + C$

13) $f(x) = \frac{1}{\sqrt{1-x^2}} - \frac{3}{1+x^2}$: $F(x) = \arcsin x - 3\arctan x + C$

15) $f(x) = (x^2+3)(x-7) = x^3 - 7x^2 + 3x - 21$: $F(x) = \frac{x^4}{4} - \frac{7x^3}{3} + \frac{3x^2}{2} - 21x + C$

17) $f(x) = x^{\frac{2}{3}} - \frac{3}{x^{\frac{1}{3}}} + 15$: $F(x) = \frac{3}{5}x^{\frac{5}{3}} - 3 \cdot \frac{3}{2}x^{\frac{2}{3}} + C = \frac{3}{5}x^{\frac{5}{3}} - \frac{9}{2}x^{\frac{2}{3}} + C$

Find $f$:

19) $f'(x) = 3x^2 - 12x + 7$, $f(1) = 1$: $f(x) = x^3 - 6x^2 + 7x + C$, $f(1) = 1 \rightarrow$ $2 + C = 1 \rightarrow$ $C = -1 \rightarrow$ $f(x) = x^3 - 6x^2 + 7x - 1$

21) $f'(x) = 5x^4 + 9x^2 + 2x$, $f(1) = 2$: $f(x) = x^5 + 3x^3 + x^2 + C$, $f(1) = 2 \rightarrow$ $5 + C = 2 \rightarrow$ $C = -3 \rightarrow$ $f(x) = x^5 + 3x^3 + x^2 - 3$

23) $f'(x) = e^x + x^2$, $f(0) = 2$: $f(x) = e^x + \frac{x^3}{3} + C$, $f(0) = 2 \rightarrow$ $1 + C = 2 \rightarrow$ $C = 1 \rightarrow$ $f(x) = e^x + \frac{x^3}{3} + 1$

25) $f'(x) = \sec^2 x - 2e^x,\ f(0) = 0:\quad f(x) = \tan x - 2e^x + C,\quad f(0) = 0 \to\quad -2 + C = 0 \to$
$C = 2 \to\quad f(x) = \tan x - 2e^x + 2$

27) $f''(x) = x - 7,\ f'(1) = 1,\ f(0) = 2:\quad f'(x) = \frac{x^2}{2} - 7x + C,\quad f'(1) = 1 \to\quad \frac{1}{2} - 7 + C =$
$1 \to\quad C = \frac{15}{2},\quad f'(x) = \frac{x^2}{2} - 7x + \frac{15}{2},\quad f(x) = \frac{x^3}{6} - \frac{7x^2}{2} + \frac{15}{2}x + C,\quad f(0) = 2 \to\quad C = 2 \to$
$f(x) = \frac{x^3}{6} - \frac{7x^2}{2} + \frac{15}{2}x + 2$

29) $f''(x) = 4x^3 + \sqrt{x} - x^3,\ f'(0) = 0,\ f''(0) = 0:\quad f'(x) = x^4 + \frac{2}{3}x^{\frac{3}{2}} - \frac{x^4}{4} + C = \frac{3x^4}{4} + \frac{2}{3}x^{\frac{3}{2}} -$
$\frac{x^4}{4} + C,\quad f'(0) = 0 \to\quad C = 0 \to\quad f'(x) = \frac{3x^4}{4} + \frac{2}{3}x^{\frac{3}{2}} - \frac{x^4}{4},\quad f(x) = \frac{3x^5}{20} + \frac{4}{15}x^{\frac{5}{2}} - \frac{x^5}{20} +$
$C,\quad f(0) = 0 \to\quad C = 0 \to\quad f(x) = \frac{3x^5}{20} + \frac{4}{15}x^{\frac{5}{2}} - \frac{x^5}{20}$

31) $f'(x) = \frac{x^2-2}{x},\ f'(1) = 0,\quad f''(x) = x - \frac{2}{x}:\quad f'(x) = \frac{x^2}{2} - 2\ln|x| + C,\quad f'(1) = 0 \to\quad \frac{1}{2} +$
$C = 0 \to\quad C = -\frac{1}{2} \to\quad f'(x) = \frac{x^2}{2} - 2\ln|x| - \frac{1}{2}$

33) $f''(x) = x^{\frac{1}{3}} - \frac{1}{x^2} - 2x,\ f'(1) = 0,\ f(1) = 0,\quad f'(x) = \frac{3}{4}x^{\frac{4}{3}} + \frac{1}{x} - x^2 + C,\quad f'(1) = 0 \to\quad \frac{3}{4} +$
$1 - 1 + C = 0 \to\quad C = -\frac{3}{4} \to\quad f'(x) = \frac{3}{4}x^{\frac{4}{3}} + \frac{1}{x} - x^2 - \frac{3}{4},\quad f(x) = \frac{3}{4} \cdot \frac{3}{7}x^{\frac{7}{3}} +$
$\ln|x| - \frac{x^3}{3} - \frac{3}{4}x + C,\quad f(1) = 0 \to\quad \frac{9}{28} - \frac{1}{3} - \frac{3}{4} + C = 0 \to\quad C = \frac{191}{276} \to\quad f(x) = \frac{9}{28}x^{\frac{7}{3}} +$
$\ln|x| - \frac{x^3}{3} - \frac{3}{4}x + \frac{191}{276}$

35) A particle is accelerating at a rate of $a(t) = t^2 - 1$. Find functions for both the velocity and position when $v(0) = 0$, and $s(0) = 10$.

$v(t) = \frac{t^3}{3} - t + v_0,\quad v(0) = 0 \to\quad v_0 = 0 \to\quad v(t) = \frac{t^3}{3} - t$

$s(t) = \frac{t^4}{12} - \frac{t^2}{2} + s_0,\quad s(0) = 10 \to\quad s_0 = 10 \to\quad s(t) = \frac{t^4}{12} - \frac{t^2}{2} + 10$

**CHAPTER 4**
**SECTION 1**
**AREA APPROXIMATION**

**EXERCISES:**

Use summation notation to rewrite the following sums:

1) $2+4+6+8+10+12=\sum_{i=1}^{6} 2i$

3) $1+3+5+9+11+13+15+17=\sum_{i=1}^{8}(2i-1)$

Rewrite the following sums without summation notation (i.e. write out all the terms):

5) $\sum_{i=1}^{3} f(x_i)=f(x_1)+f(x_2)+f(x_3)$

Use a) left endpoints and b) right endpoints to approximate the area under the curve (and above the $x$-axis) of the following functions over the given interval using 4 rectangles:

7) $f(x)=\frac{1}{x}$, $[1,5]$

a) Using left endpoints:

We observe that the area is approximated using 4 rectangles, so $n=4$ here. $\Delta x=\frac{b-a}{n}=\frac{5-1}{4}=1$. So now we have:

$\sum_{i=1}^{n} f(x_i)\Delta x= \sum_{i=1}^{4} f(x_i)\cdot 1$. Now let's write out all our terms: First we find our $x$-values:

$x_1=1,\ x_2=2,\ x_3=3,\ x_4=4$. How did we find these? $x_1$ is the first $x$-value in which the function touches the graph. This occurs at $x=1$. The second one occurs at $x=2$, etc. There are 4 rectangles, and there are 4 $x$-values here. Note that we do not have an $x$-value $=5$. (This is a common student error). We have only 4 rectangles, starting with $x=1$. (Hint, for the graph on the right, we will have an $x=5$, but no $x=1$. Again, giving us 4 rectangles).

So $\sum_{i=1}^{n} f(x_i)\Delta x= \sum_{i=1}^{4} f(x_i)\cdot 1=$

$[f(1)+f(2)+f(3)+f(4)]\cdot 1=$

$$\left[\left(\frac{1}{1}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}\right)+\left(\frac{1}{4}\right)\right]\cdot 1=\frac{25}{12}$$

b) Next, we use right endpoints. We still have 4 rectangles, so our $\Delta x$ remains the same. Now $x_1=2,\ x_2=3,\ x_3=4,\ x_4=5$. And:

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{4} f(x_i)\cdot 1 =$$
$$[f(2)+f(3)+f(4)+f(5)]\cdot 1 =$$
$$\left[\left(\frac{1}{2}\right)+\left(\frac{1}{3}\right)+\left(\frac{1}{4}\right)+\left(\frac{1}{5}\right)\right]\cdot 1 = \frac{77}{60}$$

9) $f(x) = \sin x, \ \ [0,\pi]$

a) Using left endpoints:

We observe that the area is approximated using 4 rectangles, so $n = 4$ here. $\Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4}$. So now we have:

$\sum_{i=1}^{n} f(x_i)\Delta x = \ \sum_{i=1}^{4} f(x_i)\cdot 1$. Now let's write out all our terms: First we find our $x$-values:

$x_1 = 0, \ x_2 = \frac{\pi}{4}, \ x_3 = \frac{\pi}{2}, \ x_4 = \frac{3\pi}{4}$. How did we find these? $x_1$ is the first $x$-value in which the function touches the graph. This occurs at $x = 0$. The second one occurs at $x = \frac{\pi}{4}$, etc. There are 4 rectangles, and there are 4 $x$-values here. Note that we do not have an $x$-value $= \frac{3\pi}{4}$. (This is a common student error). We have only 4 rectangles, starting with $x = 0$. (Hint, for the graph on the right, we will have an $x = \frac{3\pi}{4}$, but no $x = 0$. Again, giving us 4 rectangles).

So $\sum_{i=1}^{n} f(x_i)\Delta x = \ \sum_{i=1}^{4} f(x_i)\cdot 1 =$
$$\left[f(0)+f\left(\frac{\pi}{4}\right)+f\left(\frac{\pi}{2}\right)+f\left(\frac{3\pi}{4}\right)\right]\cdot\frac{\pi}{4} =$$
$$[\sin 0)+\left(\sin\frac{\pi}{4}\right)+\left(\sin\frac{\pi}{2}\right)+\left(\sin\frac{3\pi}{4}\right)]\cdot\frac{\pi}{4} =$$
$$\left(0+\frac{\sqrt{2}}{2}+1+\frac{\sqrt{2}}{2}\right)\cdot\frac{\pi}{4} = \frac{\sqrt{2}\pi+\pi}{4}.$$

b) Next, we use the graph on the right using right endpoints. We still have 4 rectangles, so our $\Delta x$ remains the same. Now $x_1 = 1, \ x_2 = 2, \ x_3 = 3, \ x_4 = 4$. And:

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{4} f(x_i)\cdot 1$$

Now let's write out all our terms: First we find our $x$-values:

$x_1 = \frac{\pi}{4}, \ x_2 = \frac{\pi}{2}, \ x_3 = \frac{3\pi}{4}, \ x_4 = \pi$ So $\sum_{i=1}^{n} f(x_i)\Delta x = \ \sum_{i=1}^{4} f(x_i)\cdot 1 =$
$$\left[f\left(\frac{\pi}{4}\right)+f\left(\frac{\pi}{2}\right)+f\left(\frac{3\pi}{4}\right)+f(\pi)\right]\cdot\frac{\pi}{4} =$$
$$[\left(\sin\frac{\pi}{4}\right)+\left(\sin\frac{\pi}{2}\right)+\left(\sin\frac{3\pi}{4}\right)+\sin\pi]\cdot\frac{\pi}{4} =$$
$$\left(\frac{\sqrt{2}}{2}+1+\frac{\sqrt{2}}{2}+0\right)\cdot\frac{\pi}{4} = \frac{\sqrt{2}\pi+\pi}{4}.$$

**CHAPTER 4**
**SECTION 2**
**THE DEFINITE INTEGRAL**

**EXERCISES:**

Rewrite the following limits as definite integrals:

1) $\lim_{n\to\infty} \sum_{i=1}^{n}(3x_i^5 + 11x_i^3)\Delta x,\ \ [1,2] = \int_1^2 (3x^5 + 11x^3)\, dx$

3) $\lim_{n\to\infty} \sum_{i=1}^{n}\left(e^{2x_i} + x^{\frac{1}{3}}\right)\Delta x,\ \ [0,1] = \int_0^1 \left(e^{2x} + x^{\frac{1}{3}}\right)\, dx$

Use the definition of the integral to evaluate the following: (Hint: this is the limit of the Riemann Sum):

5) $\int_0^2 (x-1)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n}(x_i - 1)\Delta x,\ \ [0,2]$

Now, we have $\Delta x = \frac{2-0}{n} = \frac{2}{n}$.

We have $x_i = a + i\Delta x = 0 + i \cdot \frac{2}{n} = \frac{2i}{n}$.

So $\lim_{n\to\infty} \sum_{i=1}^{n}(x_i - 1)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right) - 1\right] \cdot \frac{2}{n} =$

$\lim_{n\to\infty} \sum_{i=1}^{n}\left[\frac{8i}{n^2} - \frac{2}{n}\right] =$ (By sum formula 9) and limit law 1) ): =

$\lim_{n\to\infty} \sum_{i=1}^{n}\frac{8i}{n^2} - \lim_{n\to\infty} \sum_{i=1}^{n}\frac{2}{n} =$ (By sum formulas 7) and 8) ): =

$$\lim_{n\to\infty} \frac{8}{n^2}\sum_{i=1}^{n} i - \lim_{n\to\infty} \frac{2}{n} \cdot n$$

Now, we must use our summation formula for $i$:

We substitute $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ into the above expression:

$$\lim_{n\to\infty} \frac{8}{n^2}\sum_{i=1}^{n} i - \lim_{n\to\infty} \frac{2}{n} \cdot n =$$

$$\lim_{n\to\infty} \frac{8}{n^2} \cdot \left(\frac{n(n+1)}{2}\right) - 2 =$$

$$4 - 2 = 2.$$

7) $\int_0^2 (x^2 + x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^2 + x_i\right)\Delta x.$

Now, we have $\Delta x = \frac{2-0}{n} = \frac{2}{n}$.

We have $x_i = a + i\Delta x = 0 + i \cdot \frac{2}{n} = \frac{2i}{n}$.

So $\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^2 + x_i\right)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 + \left(\frac{2i}{n}\right)\right] \cdot \frac{2}{n}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{4i^2}{n^2} + \frac{2i}{n}\right] \cdot \frac{2}{n} =$$

$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{8i^2}{n^3} + \frac{4i}{n^2}\right] =$ (By sum formula 9) and limit law 1) ): =

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{8i^2}{n^3} + \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4i}{n^2} =$ (By sum formula 8) : =

$$\lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2 + \lim_{n\to\infty} \frac{4}{n^2} \sum_{i=1}^{n} i$$

Now, we must use our summation formula for $i^2$ and $i$:

We substitute $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ into the above expression, and $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2 + \lim_{n\to\infty} \frac{4}{n^2} \sum_{i=1}^{n} i$$

$$\lim_{n\to\infty} \frac{8}{n^3} \cdot \left(\frac{n(n+1)(2n+1)}{6}\right) + \lim_{n\to\infty} \frac{4}{n^2} \left(\frac{n(n+1)}{2}\right) =$$

$$\frac{16}{6} + \frac{4}{2} = \frac{14}{3}.$$

9) $\int_1^2 (x^2 - x + 1)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^2 - x_i + 1\right)\Delta x.$

Now, we have $\Delta x = \frac{2-1}{n} = \frac{1}{n}.$

We have $x_i = a + i\Delta x = 1 + i \cdot \frac{1}{n} = 1 + \frac{i}{n}.$

So $\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^2 - x_i + 1\right)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1 + \frac{i}{n}\right)^2 - \left(1 + \frac{i}{n}\right) + 1\right] \cdot \left(\frac{1}{n}\right)$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[1 + \frac{2i}{n} + \frac{i^2}{n^2} - 1 - \frac{i}{n} + 1\right] \cdot \left(\frac{1}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{i^2}{n^2} + \frac{i}{n} + 1\right] \cdot \left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{i^2}{n^3} + \frac{i}{n^2} + \frac{1}{n}\right] =$$

$=$ (By sum formula 9) and limit law 1): =

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} + \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2} + \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} =$ (By sum formulas 7) and 8)) : =

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 + \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i + \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} 1$$

Now, we must use our summation formula for $i^2$ and $i$:

We substitute $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ into the above expression, and $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 + \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i + \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} 1 =$$

$$\lim_{n\to\infty} \frac{1}{n^3} \cdot \left(\frac{n(n+1)(2n+1)}{6}\right) + \lim_{n\to\infty} \frac{1}{n^2} \left(\frac{n(n+1)}{2}\right) + \lim_{n\to\infty} \frac{1}{n} \cdot n =$$

$$\frac{2}{6} + \frac{1}{2} + 1 = \frac{11}{6}.$$

Evaluate the following integrals by combining the properties for definite integrals along with your knowledge of areas:

11) $\int_1^2 (2x+1)\,dx$ (Hint: You should get the same answer you did in number 6) with less work!) $= 2\int_1^2 x\,dx + \int_1^2 1\,dx = \frac{1+2}{2} \cdot 2\ (\text{area of a trapezoid}) + 1 \cdot (2 - 1)\ (\text{area of a rectangle}) = 3 + 1 = 4$

Use properties of definite integrals to evaluate the integrals:

13) If $\int_0^2 f(x)\,dx = 5,$ and $3\int_2^4 f(x)\,dx = 9$, what is $\int_0^4 f(x)\,dx?$ $5 + \frac{9}{3} = 8$

**CHAPTER 4**
**SECTION 3**
**THE FUNDAMENTAL THEOREM OF CALCULUS**

**EXERCISES:**

Evaluate the following integrals using the Fundamental Theorem of Calculus:

1) $\int_0^1 (x^2 - 4x + 7)\,dx = \left(\frac{x^3}{3} - 2x^2 + 7x\right)\Big]_0^1 = \left(\frac{1}{3} - 2 + 7\right) - 0 = \frac{16}{3}$

3) $\int_1^2 (4x^3 - 3x^2 + x - 1)\,dx = (x^4 - x^3 - x)]_1^2 = (16 - 8 - 2) - (1 - 1 - 1) = 7$

5) $\int_{-1}^1 x^{\frac{1}{3}}\,dx = \frac{3}{4}x^{\frac{4}{3}}\Big]_{-1}^1 = \frac{3}{4}(1 - 1) = 0$

7) $\int_0^2 (x-2)(x+3)\,dx = \int_0^2 (x^2 + x - 6)dx = \left(\frac{x^3}{3} + \frac{x^2}{2} - 6x\right)\Big]_0^2 = \frac{8}{3} + 2 - 12 = -\frac{22}{3}$

9) $\int_0^\pi (\sin x + \sec^2 x + e^x)\,dx = (-\cos x + \tan x + e^x)]_0^\pi = (-\cos\pi + \tan\pi + e^\pi) - (-\cos 0 + \tan 0 + e^0) = (1 + 0 + e^\pi) - (-1 + 0 + 1) = e^\pi + 1$

11) $\int_1^2 \frac{x^2 - 2x + x^{-1}}{x}\,dx = \int_1^2 (x - 2 + x^{-2})\,dx = \left(\frac{x^2}{2} - 2x - \frac{1}{x}\right)\Big]_1^2 = \left(2 - 4 - \frac{1}{2}\right) - \left(\frac{1}{2} - 2 - 1\right) = -4$

13) $\int_0^1 3^x\,dx = 3^x \ln 3]_0^1 = 3\ln 3 - \ln 3 = \ln 9$

15) $\int_0^{\frac{\pi}{4}} (\cos x + \sec x \tan x)\,dx = \sin x + \sec x]_0^{\frac{\pi}{4}} = (\sin\frac{\pi}{4} + \sec\frac{\pi}{4}) - (\sin 0 + \sec 0) = \frac{\sqrt{2}}{2} + \sqrt{2} - 1$

17) $\int_0^1 \left(\frac{1}{1+x^2} - 2x\right)\,dx = (\arctan x - x^2)]_0^1 = \arctan 1 - 1 - (\arctan 0 - 0) = \frac{\pi}{4} - 1$

19) $\int_0^1 (x^2 + 2^x)\,dx = \frac{x^3}{3} + 2^x \ln 2\Big]_0^1 = \frac{1}{3} + 2\ln 2 - \ln 2 = \frac{1}{3} + \ln 2$

Find the area of the following functions over the given interval: (Hint: for some problems, you may have to find the interval from the information given).

21) $y = 2x^3$, $[0,1]$: $A = \int_0^1 2x^3\,dx = \frac{x^4}{2}\Big]_0^1 = \frac{1}{2}$

23) $y = \cos x + x^2, \ \left[0, \frac{\pi}{4}\right]$: $A = \int_0^{\frac{\pi}{4}} (\cos x + x^2)\, dx = \left(\sin x + \frac{x^3}{3}\right]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\pi^3}{192}$

25) $y = x^3, \ y = 0, \ x = 2$: $A = \int_0^2 x^3\, dx = \frac{x^4}{4}\Big]_0^2 = 4$

Find the derivative of the function:

27) $F(x) = \int_2^x \sqrt{3t^2 + 4t}\, dt = \sqrt{3x^2 + 4x}$

29) $F(x) = \int_\pi^x \frac{\ln t}{\sqrt{t}}\, dt = \frac{\ln x}{\sqrt{x}}$

$F(x) = \int_2^{\sqrt{x}} (2t - e^t)\, dt$: We will do a substitution: Let $u = \sqrt{x}$. We next take the derivative: $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$.

So we get $\frac{d}{dx}\left(\int_2^{\sqrt{x}} (2t - e^t)\, dt\right) = \frac{d}{du}\left(\int_2^u (2t - e^t)\, dt\right) \cdot \frac{du}{dx}$ (Chain Rule) $= (2u - e^u) \cdot \frac{du}{dx} = (2u - e^u) \cdot \frac{1}{2\sqrt{x}} = \left(2\sqrt{x} - e^{\sqrt{x}}\right) \cdot \frac{1}{2\sqrt{x}} = 1 - \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

33) $F(x) = \int_{x^2}^x \sin t\ dt$ (Hint: Rewrite using two of the properties of definite integrals).

$= -\int_0^{x^2} \sin t\, dt + \int_0^x \sin t\, dt =$ Let $u = x^2$. For the first integral: We next take the derivative: $\frac{du}{dx} = 2x$.

So we get $\frac{d}{dx}\left(-\int_1^{x^2} \sin t\ dt\right) = \frac{d}{du}\left(-\int_1^u \sin t\ dt\right) \cdot \frac{du}{dx}$ (Chain Rule) $= -\sin u \cdot \frac{du}{dx} = -(\sin u) \cdot 2x = -2x \sin x^2$. So the final answer is $-2x \sin x^2 + \sin x$.

**CHAPTER 4**
**SECTION 4**
**INDEFINITE INTEGRALS AND THE NET CHANGE THEOREM**

**EXERCISES:**

Find the general indefinite integral:

1) $\int (3x^2 - 4x + 8)\,dx = x^3 - 2x^2 + 8x + C$

3) $\int (5x^4 - 2x^2 + 12x)\,dx = x^5 - \frac{2}{3}x^3 + 6x^2 + C$

5) $\int (x-1)(x+3)\,dx = \int (x^2 + 2x - 3)\,dx = \frac{x^3}{3} + x^2 - 3x + C$

7) $\int \left(2\sqrt{x} - \frac{3}{x} + x^{-2}\right)\,dx = \frac{4}{3}x^{\frac{3}{2}} - 3\ln|x| - \frac{1}{x} + C$

9) $\int (2e^x + \sec^2 x)\,dx = 2e^x + \tan x + C$

11) $\int \left(\sec x \tan x - \frac{1}{1+x^2} + x^{\frac{1}{3}}\right)\,dx = \sec x - \arctan x + \frac{3}{4}x^{\frac{4}{3}} + C$

13) $\int \left(\frac{x^{\frac{1}{2}} - 2x + 7}{x}\right)\,dx = \int \left(x^{-\frac{1}{2}} - 2 + \frac{7}{x}\right)\,dx = 2x^{\frac{1}{2}} - 2x + 7\ln|x| + C$

15) $\int \left(\frac{1}{\sqrt{1-x^2}} + \sec^2 x + \frac{2}{x}\right)\,dx = \arcsin x + \tan x + 2\ln|x| + C$

17) A particle moves in a straight line such that its velocity (as a function of time $= t$ is $v(t) = t^2 - 5t - 6$. (In meters/second).

   a) Find the displacement for $0 \le t \le 10$
   b) Find the total distance traveled during this time.

   a) Displacement is $\int_0^{10} (t^2 - 5t - 6)\,dt = \left(\frac{t^3}{3} - \frac{5t^2}{2} - 6t\right)\Big|_0^{10} = \frac{1000}{3} - 250 - 60 = \frac{70}{3}$. This means the particle ends up $\frac{70}{3}$ meters to the right of where it started (after $10\ s$), since the displacement is positive.

   b) The total distance traveled: $v(t) = t^2 - 5t - 6 = (t-6)(t+1)$, implies $v(t) \le 0$ on $[0,6]$ and $v(t) \ge 0$ on $[6,10]$.

So distance $= \int_0^6 -(t^2 - 5t - 6)\, dt + \int_6^{10} (t^2 - 5t - 6)\, dt = \left(-\frac{t^3}{3} + \frac{5t^2}{2} + 6t\right)\Big|_0^6 + \left(\frac{t^3}{3} - \frac{5t^2}{2} - 6t\right)\Big|_6^{10} = -72 + 90 + 36 + \frac{1000}{3} - 250 - 60 - (72 - 90 - 36) = \frac{394}{3}$ meters traveled.

**CHAPTER 4**
**SECTION 5**
**THE SUBSITUTION RULE FOR INTEGRALS**

**EXERCISES:**

Use U-substitution to find the following general indefinite integrals:

1) $\int (x^2-7)^8\, 2x\, dx$: Choose $u = x^2 - 7$.

$du = 2x\, dx \rightarrow \quad dx = \frac{du}{2x}$.

Substitute: $\int (x^2-7)^8\, 2x\, dx = \int u^8 \cdot 2x \cdot \frac{du}{2x} = \int u^8\, du = \frac{u^9}{9} + C = \frac{(x^2-7)^9}{9} + C$

3) $\int (3x^2+9)^3\, x\, dx$: Choose $u = 3x^2 + 9$.

$du = 6x\, dx \rightarrow \quad dx = \frac{du}{6x}$.

Substitute: $\int (3x^2+9)^3\, x\, dx = \int u^3 \cdot x \cdot \frac{du}{6x} = \frac{1}{6}\int u^3\, du = \frac{u^4}{24} + C = \frac{(3x^2+9)^4}{24} + C$

5) $\int (x^3-10)^{\frac{1}{3}}\, 2x^2\, dx$: Choose $u = 3x^2 + 9$.

$du = 6x\, dx \rightarrow \quad dx = \frac{du}{6x}$.

Substitute: $\int (3x^2+9)^3\, x\, dx = \int u^3 \cdot x \cdot \frac{du}{6x} = \frac{1}{6}\int u^3\, du = \frac{u^4}{24} + C = \frac{(3x^2+9)^4}{24} + C$

7) $\int \sqrt{x^3+7}\, x^2\, dx$: Choose $u = x^3 + 7$.

$du = 3x^2\, dx \rightarrow \quad dx = \frac{du}{3x^2}$.

Substitute: $\int \sqrt{x^3+7}\, x^2\, dx = \int u^{\frac{1}{2}} \cdot x^2 \cdot \frac{du}{3x^2} = \frac{1}{3}\int u^{\frac{1}{2}}\, du = \frac{2}{3} \cdot \frac{u^{\frac{3}{2}}}{24} + C = \frac{(x^3+7)^{\frac{3}{2}}}{36} + C$

9) $\int (x^3 - x^2 + 3)^7 \, (3x^2 - 2x)\, dx$:  : Choose $u = x^3 - x^2 + 3$.

$du = (3x^2 - 2x)\, dx \rightarrow \quad dx = \dfrac{du}{3x^2 - 2x}$.

Substitute: $\int (x^3 - x^2 + 3)^7 \, (3x^2 - 2x)\, dx = \int u^7 \cdot du = \frac{u^8}{8} + C = \frac{(x^3 - x^2 + 3)^8}{8} + C$

11) $\int (x^2 - 2x)^{10} \, (x - 1)\, dx$:  Choose $u = x^2 - 2x$.

$du = (2x - 2)\, dx \rightarrow \quad dx = \dfrac{du}{2x - 2}$.

Substitute: $\int (x^2 - 2x)^{10} \, (x - 1)\, dx = \frac{1}{2} \int u^{10} \cdot du = \frac{u^{11}}{22} + C = \frac{(x^2 - 2x)^{11}}{22} + C$

13) $\int \frac{3x^2 + 4x + 1}{x^3 + 2x^2 + x - 1}\, dx$:  Choose $u = x^3 + 2x^2 + x - 1$.

$du = (3x^2 + 4x + 1)\, dx \rightarrow \quad dx = \dfrac{du}{3x^2 + 4x + 1}$.

Substitute: $\int \frac{3x^2 + 4x + 1}{x^3 + 2x^2 + x - 1}\, dx = \int \frac{1}{u} \cdot du = \ln|u| + C = \ln|x^3 + 2x^2 + x - 1| + C$

15) $\int \frac{6x + 9}{\sqrt{x^2 + 3x}}\, dx$:  Choose $u = x^2 + 3x$.

$du = (2x + 3)\, dx \rightarrow \quad dx = \dfrac{du}{2x + 3}$.

Substitute: $\int \frac{6x + 9}{\sqrt{x^2 + 3x}}\, dx = 3 \int u^{-\frac{1}{2}} \cdot du = 6\sqrt{u} + C = 6\sqrt{x^2 + 3x} + C$

17) $\int \frac{1}{3}\, e^{-2x}\, dx$:  Choose $u = -2x$.

$du = -2\, dx \rightarrow \quad dx = \dfrac{du}{-2}$.

Substitute: $\int \frac{1}{3}\, e^{-2x}\, dx = -\frac{1}{6} \int e^u \cdot du = -\frac{1}{6} e^u + C = -\frac{1}{6} e^{-2x} + C$

19) $\int e^{3x^2}\, 4x\, dx$:   Choose $u = 3x^2$.

$du = 6x^2\, dx \rightarrow \quad dx = \dfrac{du}{6x^2}.$

Substitute: $\int e^{3x^2}\, 4x\, dx = \frac{2}{3}\int e^u \cdot du = \frac{2}{3}e^u + C = \frac{2}{3}e^{3x^2} + C$

21) $\int (3 - e^x)(3 - e^x)\, dx = \int (9 - 6e^x + e^{2x})\, dx = 9x - 6e^x + \frac{1}{2}e^{2x} + C$ $\Big($Last term used u − subsitution with $u = 2, du = 2\, dx, \int e^{2x} dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^{2x} + C\,\Big)$

23) $\int \sin 5x\;\, dx$: Choose $u = 5x$.

$du = 5\, dx \rightarrow \quad dx = \dfrac{du}{5}.$

Substitute: $\int \sin 5x\;\, dx = \frac{1}{5}\int \sin u\, du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin 5x + C$

25) $\int \cot x\;\, dx = \int \frac{\cos x}{\sin x}\, dx$: Choose $u = \sin x$.

$du = \cos x\;\, dx \rightarrow \quad dx = \dfrac{du}{\cos x}.$

Substitute: $\int \frac{\cos x}{\sin x}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln|\sin x| + C$

27) $\int \sec^2 x \tan x\, dx$: Choose $u = \tan x$.

$du = \sec^2 x\;\, dx \rightarrow \quad dx = \dfrac{du}{\sec^2 x}.$

Substitute: $\int \sec^2 x \tan x\, dx = \int u\;\, du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C$

29) $\int \sec^2 x \tan^2 x\, dx$: Choose $u = \tan x$.

$du = \sec^2 x\;\, dx \rightarrow \quad dx = \dfrac{du}{\sec^2 x}.$

Substitute: $\int \sec^2 x \tan^2 x\, dx\;\, dx = \int u^2\;\, du = \frac{u^3}{3} + C = \frac{\tan^3 x}{3} + C$

31) $\int (\sin x + 1)(\cos x + 2)\, dx = \int (\sin x \cos x + 2\sin x + \cos x + 2)\, dx = \int (\sin x \cos x)\, dx + 2\int \sin x\, dx + \int \cos x\, dx + \int 2\, dx = \frac{(\sin^2 x)}{2}$ (using $u = \sin x$, $du = \cos x\, dx$) $-$ $2\cos x + \sin x + 2x + C$

33) $\int \frac{\arctan x}{1+x^2}\, dx$: Choose $u = \arctan x$.

$$du = \frac{1}{1+x^2}\, dx \rightarrow \quad dx = (1+x^2)\, du.$$

Substitute: $\int \frac{\arctan x}{1+x^2}\, dx = \int u\ du = \frac{u^2}{2} + C = \frac{(\arctan x)^2}{2} + C$

35) $\int \frac{1}{4+x^2}\, dx$ (Hint: Factor the 4 out of the denominator, then do the u-substitution):

$= \frac{1}{4}\int \left(\frac{1}{1+\left(\frac{x}{2}\right)^2}\right) dx.$ Choose $u = \frac{x}{2}$.

$$du = \frac{1}{2}\, dx \rightarrow \quad dx = 2\, du.$$

Substitute: $\int \frac{1}{4+x^2}\, dx = \frac{1}{2}\int \frac{1}{1+u^2} \cdot du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan\frac{x}{2} + C$

37) $\int \sqrt{x-1}\, x^2\, dx$:

This one involves a little trick: Choose $u = x - 1 \rightarrow \quad du = dx$.

Okay, so far so good, right? But what about the $x^3$? It's not going to cancel.

The trick: If $u = x - 1$, then $x = u + 1$.

So, $\int \sqrt{x-1}\, x^3\, dx = \int u^{\frac{1}{2}}(u+1)^3\, du = \int u^{\frac{1}{2}}(u^3 - 3u^2 + 3u - 1)\, du = \int \left(u^{\frac{7}{2}} - 3u^{\frac{5}{2}} + 3u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du.$

Now, we have a form that we can integrate:

$$\int\left(u^{\frac{7}{2}}-3u^{\frac{5}{2}}+3u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)=\frac{2}{9}u^{\frac{9}{2}}-\frac{6}{7}u^{\frac{7}{2}}+\frac{6}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}+C$$

$$=\frac{2}{9}(x-1)^{\frac{9}{2}}-\frac{6}{7}(x-1)^{\frac{7}{2}}+\frac{6}{5}(x-1)^{\frac{5}{2}}-\frac{2}{3}(x-1)^{\frac{3}{2}}+C.$$

Evaluate the definite integral (Hint: Use U-substitution):

39) $\int_0^1 (4x^3-9)^4\, 12x^2\, dx$:

We choose $u = 4x^3 - 9$

When $x = 0, u = -9,$ and when $x = 1,\ u = -5$

$du = 12x^2 dx \rightarrow \quad dx = \dfrac{du}{12x^2}$

$$\int_{-9}^{-5} u^4 du = \frac{u^5}{5}\Big]_{-9}^{-5} = \frac{(-5)^5}{5} - \frac{(-9)^5}{5} = \frac{55924}{5}$$

41) $\int_0^3 e^{2x}\, dx$:

We choose $u = 2x$

When $x = 0, u = 0,$ and when $x = 3,\ u = 6$

$du = 2dx \rightarrow \quad dx = \dfrac{du}{2}$

$$\int_0^3 e^{2x}\, dx = \int_0^6 e^u du = e^u]_0^6 = e^6 - e^0 = e^6 - 1$$

43) $\int_0^1 \frac{2x-3}{x^2-3x+7}\, dx$:

We choose $u = x^2 - 3x + 7$

When $x = 0, u = 7,$ and when $x = 1,\ u = 5$

$du = (2x-3)dx \rightarrow \quad dx = \dfrac{du}{(2x-3)}$

$$\int_0^1 \frac{2x-3}{x^2-3x+7}\,dx = \int_7^5 \frac{1}{u}\,du = -\int_5^7 \frac{1}{u}\,du = -(\ln 7 - \ln 5) = -\ln\left(\frac{7}{5}\right) = \ln\left(\frac{5}{7}\right)$$

45) $\int_0^{\frac{\pi}{4}} \sec x \tan x\, dx$:

We don't need a u-substitution here: $\int_0^{\frac{\pi}{4}} \sec x \tan x\, dx = \sec x\Big]_0^{\frac{\pi}{4}} = \sqrt{2} - 1$

47) $\int_0^{\pi} \cos^3 x \sin x\, dx$:

We choose $u = \cos x$

When $x = 0, u = 1,$ and when $x = \pi,\ u = -1$

$du = -\sin x\, dx \rightarrow \quad dx = -\dfrac{du}{\sin x}$

$\int_0^{\pi} \cos^3 x \sin x\, dx = -\int_1^{-1} u^3 du = \int_{-1}^{1} u^3\, du = 0,$ because it is an odd function.

49) $\int_0^1 \frac{2 \arctan x}{1+x^2}\, dx$:

We choose $u = \arctan x$

When $x = 0, u = 0,$ and when $x = 1,\ u = \frac{\pi}{4}$

$du = \dfrac{1}{1+x^2} dx \rightarrow \quad dx = (1+x^2)\, du$

$\int_0^1 \frac{2 \arctan x}{1+x^2}\, dx = 2\int_0^{\frac{\pi}{4}} u\, du = u^2\Big]_0^{\frac{\pi}{4}} = \frac{\pi^2}{16}$

51) $\int_0^1 (e^{3x} - \sin x \cos x)\, dx$ (Hint: Use a property of integrals to rewrite as two integrals):

$$= \int_0^1 e^{3x} dx - \int_0^1 \sin x \cos x\, dx:$$

$$= \frac{1}{3}\int_0^3 e^u du = \frac{1}{3}(e^3 - 1) - \int_0^{\sin 1} u\, du = \frac{1}{3}(e^3-1) - \frac{u^2}{2}\Big]_0^{\sin 1} = \frac{1}{3}(e^3-1) - \frac{\sin^2 1}{2}$$

53) $\int_2^3 \frac{e^x}{e^x - e}\, dx$:

We choose $u = e^x - e$

When $x = 2, u = e^2 - e,$ and when $x = 3,\ u = e^3 - e$

$du = e^x\,dx \rightarrow \quad dx = \dfrac{du}{e^x}$

$$\int_2^3 \frac{e^x}{e^x - e}\,dx = \int_{e^2-e}^{e^3-e} \frac{1}{u}\,du = \ln u\Big]_{e^2-e}^{e^3-e} = \ln(e^3 - e) - \ln(e^2 - e) \approx 17.37 - 4.67 = 12.7$$

Use symmetry to evaluate the following integrals more efficiently:

55) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = 2\int_0^{\frac{\pi}{2}} \cos x\,dx = 2\sin x\Big]_0^{\frac{\pi}{2}} = 2$

57) $\int_{-100}^{100} (x^3 + x)\,dx = 0$, due to symmetry, because it is odd.