CALCULUS ONE OVER COFFEE

A Textbook and User-Friendly Guide to Calculus One

PREFACE

The purpose of this book was originally meant to be a companion to a traditional Calculus textbook. It was not meant to replace a formal textbook, while being a bit more than just a book of problem sets. Its aim was to clarify some difficult concepts, and to provide some easier exercises in addition to the traditional classroom textbook. After I starting writing it, I decided to make it into an actual textbook. It can still serve as a companion to a different text, but it can also perform as a standalone text as well. After teaching Calculus for close to two decades, I have found most students struggle with the typical formalism in a standard textbook. My students have been looking for more user-friendly language along with some easier warm-up exercises. My hope is that this text will provide that gap for students. This book has all the formal definitions, and proofs as necessary, but it also includes more user-friendly language, and a bit of an informal attitude along with the standard traditional formalism. Its original title was A Calculus Companion, which I later changed to Calculus One Over Coffee. I feel that it is more than a companion text, but the new title still conveys a more casual (and hopefully more understandable) approach to Calculus One.

CHAPTER 0

CALCULUS MOTIVATION

It is my understanding and observation from pedagogical research and experience that for students to master a subject, they must be properly motivated.

In this chapter, we will explore what motivated the invention of Calculus along with some practical applications. I believe by understanding the history and practical uses for the subject, we will excite and motivate most students to want to study it in its entirety.

We start with the history of Calculus invented by Sir Isaac Newton (along with Gottfried Wilhelm Leibnitz). Sir Isaac Newton was born in England in 1642 to the yeoman class. He lived from 1642-1727. During his life he witnessed the Civil Wars and Interregnum, along with the Restoration of the Stuarts and the Glorious Revolution.

Descartes had great influence over Newton. It involved mathematics and nature, though Newton disagreed with many of Descartes's views such as areas of epistemology and the theory of matter.

Newton had an interest in chemistry including alchemy (a subject in extra-natural substance), ancient history, and Christianity.

Newton believed "the prime of my age for invention" was during 1665 and 1666. In mechanics, there was an entry dated January 20, 1664 in his "Waste Book". The "Waste Book" was a book that contained many of his early mathematics. He developed a comprehensive mathematical theory about derivatives and tangents (rates of change that we will discuss soon), along with integrals (areas under curves also to be discussed).

It is believed that in 1684-1685, Newton abandoned the tract De motu, and started the *Principia*. The formal title is "*Philosophiae Naturalis Principia Mathematica* (Latin), or *Mathematical Principles of Natural Philosophy* (English), which we frequently refer to as the *Principia*. This work was 3 books, first published in July 1687. They included Newton's laws of motion, and formed the basis for classical mechanics, Newton's law of universal gravitation, and Kepler's laws of planetary motion. The language we are used to, and that will be developed in the text, were largely absent in this work. The mathematics in the *Principia* were less classical than Newton cared to admit. In response to confused students, he translated the geometry into algebraic language for them.

He was knighted in 1705 by Queen Anne. It is believed to be more politically based, than it was based on his scientific achievements.

Newton was also a clever illustrator including some geometrical drawings, and illustrations from Sacred Scriptures referred to as "mathematical magic"!

Newton died in his sleep in March, 1727. Mercury was found in his hair after his death. Voltaire attended his funeral, who called it the funeral of a king who had done well by his subjects.

Gottfried Wilhelm Leibnitz was born in Leipzig on July 3, 1646. He lived from 1646-1716.

Leibnitz was a philosopher who was known as the "man of principles". He was known for a list of 10 principles. Leibnitz was known to be more of a philosopher with Newton being more of a scientist, and they were at odds

He had an early work called *Confessio natuae contra Atheistas*, defending "God's cause". He defended Christian doctrine with his idea of a natural theology.

Leibnitz and Newton had a great calculus controversy. In 1699, the discovery of calculus became an open controversy. In 1685, a mathematician named John Wallis who wrote *Algebra*, had information extracted from Newton's *First Letter* to Leibnitz in June of 1676. Wallis also had a copy of the *Second Letter.* In 1692, Newton sent Wallis information on the method of fluxions (which included problems of tangents, extrema, and quadratures of curves), which Wallis printed in his *Algebra.*

Nine years after Leibnitz, and eight years after John Craige (one of the first British mathematicians who sought out Newton at Cambridge), Newton's calculus was now before the world.

Newton's work did not differ from differential calculus, except he called a differential a fluxion; and he called an integral a fluent. Johann Bernoulli (Leibnitz's friend) told Leibnitz that he did not know if Newton may have fabricated his own method after seeing his friend's (Leibnitz's) calculus.

Leibnitz's reaction to the *Principia* was that it was outstanding in regards to its quantitative analysis of physical forces. He began to compose a different, neo-Cartesian account of orbital forces. He wrote a paper called "Tentamen de motuum coelestium causis", which may have been an attempt to invalidate the reasoning in the *Principia* and its concepts of real forces.

Newton suspected Leibnitz had lied, and that his theory was a "mirror-image" of his own work. He thought of Leibnitz's physics as contorted, redundant, and geometrically unsound.

Around 1699, after Leibnitz printed the third volume of *Calculus Differentials'*, John Wallis noticed that the Leibnitzian school of mathematics was moving ahead of the Newtonians. The Bernoulli brothers (Johann and Jakob) were not only brilliant mathematicians, but also big promotors of Leibnitz's calculus.

The *Principia* gave Newton a great reputation. Twenty years later, Newton had the intellectual respect of all Europe, and the idea that he had stolen his calculus ideas from Leibnitz were barely mentioned.

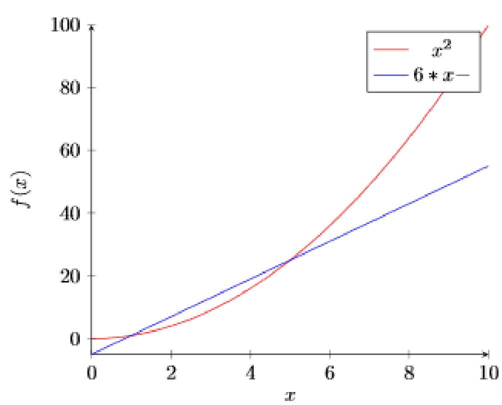
Let us now discuss some practical applications for Calculus. We see Calculus was invented to better understand the world and its behavior. We understand that it explains very precisely the laws of motion, and allows us to both measure and make predictions about our surroundings.

We will now discuss three practical and important applications of Calculus:

1) Velocity and Acceleration
2) How to maximize Profit
3) How to find the area under a curve

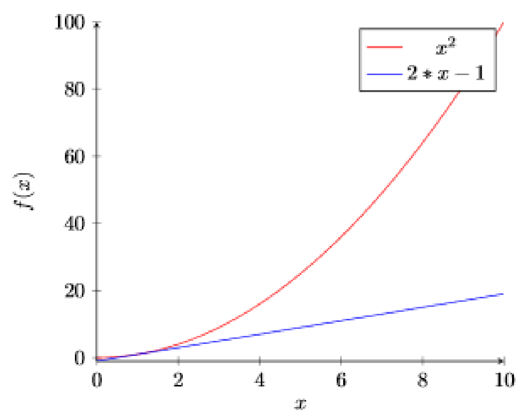
First, we visit the velocity problem. Velocity is a common problem, and should be familiar to most readers. We commonly think of velocity as speed, whereas speed is actually the absolute value of velocity. Velocity can be positive or negative, depending on one's direction. We understand velocity to be the rate of change of distance. We will first consider average velocity, which will be a linear graph with a constant slope, i.e. $v = \frac{\Delta d}{\Delta t}$, where v is the slope of the line that connects two points on a distance vs time graph. All of this is well and good, as you have already learned how to calculate an average rate of change of a function in your algebra class. You may ask, what does all of this have to do with Calculus? Notice that all we have talked about thus far are average rates of change. Calculus will add to what we know about rates of change, giving us information about instantaneous rates of change. For example, let's go back to our velocity problem: Using Calculus, we will be able to determine our instantaneous velocity at any time that we are interested in, under certain conditions that will be fleshed out in great detail in a future section. Next, we can think about the rate of change of velocity, which will give us acceleration. We will use similar methods to obtain information about instantaneous acceleration using similar methods of Calculus again.

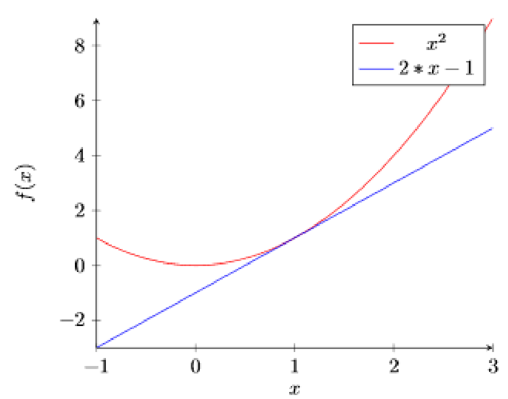
Let us observe the graph below. We will assume the graph to be a graph of distance vs time. Let us now calculate the average velocity between two different points in time. We choose the average velocity between $t = 1\ second$ to $t = 5\ seconds$: (We let x respresent time, and $f(x)$ respresent distance).



We observe the slope of the blue line that connects the times $x = 1$ and $x = 5$ represents the average velocity of this distance graph between $t = 1s$ and $t = 5s$. $v = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5^2 - 1^2}{5 - 1} = \frac{24}{4} = 6\frac{units}{second}$. This should be familiar from your Algebra class.

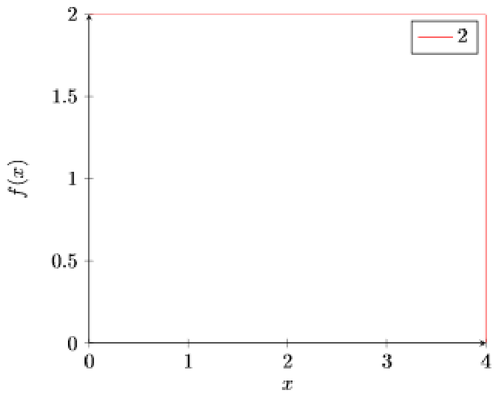
Now let's look at the graph below. The slope of the blue line now represents the instantaneous velocity of the function at $t = 1\ second$. We will need Calculus to derive this result. (2 different views).

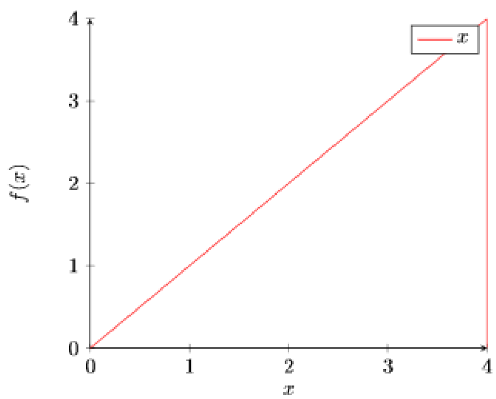




Perhaps you are not interested in Physics in general, or velocity in particular. But, let's say, you are interested in Business. Differential Calculus can also be used to find out how to maximize your profit given an appropriate function. Both maximization and velocity are applications of Differential Calculus.

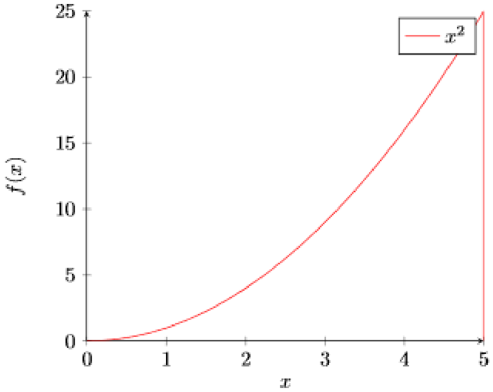
Thirdly, we might want to find the area under the curve of a graph. We can already easily do this for functions like $y = 2$, which is simply the area of a rectangle, bh; or for a function like $y = x$, which is the area of a triangle, $\frac{1}{2}bh$. See the graphs below.





We can easily calculate the area under the curve (and above the x-axis) of these functions using basic geometry.

But now let's look at another fairly simple function: $f(x) = x^2$ from $x = 0$ to $x = 5$:



This function is rather simple, but look at the difficulty of calculating the area under this curve (and above the x-axis). With our knowledge of geometry, this will be quite difficult. We will need Calculus in order to accomplish this goal.

Some practical applications of finding the area under a curve might be to find distance from knowing velocity. (We already discussed velocity from distance – it was differential calculus). Other applications for finding the area under a curve could be, e.g., the work done by an object. You can also find the total profit, if you know the marginal profit of a function. These three applications are applications of Integral Calculus, which is related to the area problem.

EXERCISES:

1) Find the average rates of change for the following functions: (i.e. in this case the average velocity):

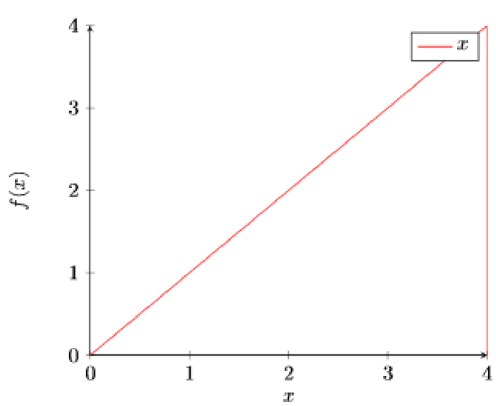
   a) $s(t) = t^3 + 5t^2 - 1, for\ t = 1\ to\ t = 5$:

   b) $s(t) = 2sint\ for\ t = 0\ to\ t = \frac{3\pi}{4}$:

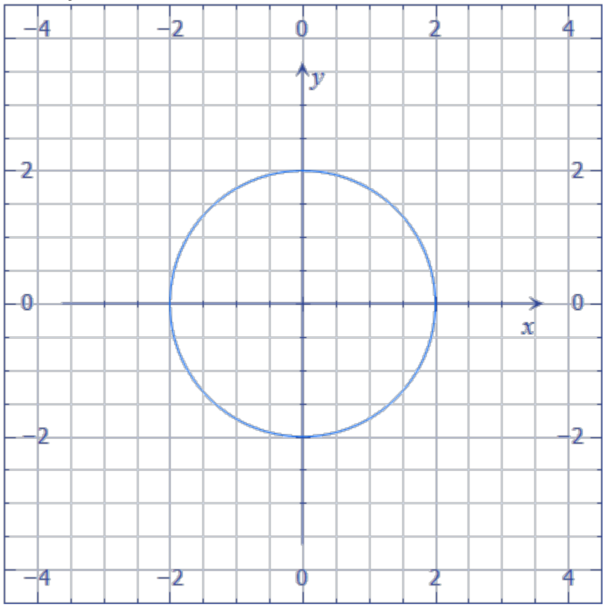
2) Find the velocity of the following function when $t = 2\ seconds$ for the function $v(t) = 3t^2 - 4$:

3) Find the area of the following objects:

   a)



b) $x^2 + y^2 = 4$



4) Find the total Cost, Revenue and Profit if the total cost is given by $C(x) = 5x + 1$, the total Revenue is given by $R(x) = 0.5x^2 + 10x - 1$ when you have produced 50 items.

CHAPTER 1
SECTION 1

LIMITS

Before we begin with an explanation of limits and the formalism that ensues, let us motivate why we would need such a concept. Without the concept and formalism of a limit, we cannot ever develop the ideas needed to solve the problems in Chapter 0.

Let us think about what we think of the word Limit. It should invoke getting close to something.

We will start with a simple story. Keep in mind that this is not a precise or formal explanation for a limit. It merely serves to get your mind thinking in this direction.

I will start with you thinking about the building you are in. In reality, there are an infinite number of paths you could take to reach the building. (This would be a Calculus 3 problem). So for our purposes, imagine you can only approach the building from two paths, from the right or from the left. Let's say that either path you take leads you to the front door. You can never reach the door, but instead, become as close as you can without touching it. Since both paths lead to the front door, we will informally call that the limit. But let's imagine that the right path leads to the front door, and the left path leads to the back door. We will say the limit does not exist, because each path led you to a different place. Now let's imagine the left path cannot lead you close to the building. Imagine a giant trench that cannot be crossed or traversed around to get to the building. The limit in this case also does not exist, because the left approach does not exist.

One important fact to remember when evaluating limits is that you never reach the value, you just get arbitrarily (or infinitesimally) close, as close as you can possibly get without touching the value.

Let us now consider the intuitive definition of a limit. (We will study the precise definition of a limit in Section 3).

$$\lim_{x \to a} f(x) = L.$$

This reads the limit as x approaches a of $f(x)$ equals L.

What this means is that we can make $f(x)$ as close as we want to L by making x as close to a as we need to, i.e. arbitrarily close to a on either side of a without ever touching a.

This may look unfamiliar to you. In fact, if you have not had Calculus, the whole idea of the limit may seem foreign to you.

Something to be noted here: a is the x-value, and L is the y-value. This may seem obvious from the definition, but I have found many students struggle with this concept.

Let us now define right-handed and left-handed limits. This will introduce more unfamiliar notation.

$$\lim_{x \to a^-} f(x) = L$$

Is the left-handed limit, and it reads the limit as x approaches a from the left of $f(x)$ equals L. What this means is that we can make $f(x)$ as close as we want to L by making x as close to a as we need to, i.e. arbitrarily close to a from the **left**, where $x < a$.

Notice that there is a minus sign as a superscript on a. This has nothing to do with the sign of the a. It merely indicates which direction you are coming from.

$$\lim_{x \to a^+} f(x) = L$$

Is the right-handed limit, and it reads the limit as x approaches a from the right of $f(x)$ equals L. This means is that we can make $f(x)$ as close as we want to L by making x as close to a as we need to, i.e. arbitrarily close to a from the right, where $x > a$.

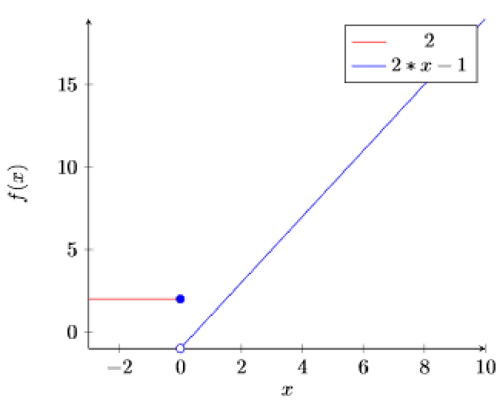
Another thing that students frequently misunderstand is that the ± sign indicates which direction you are coming **from**, not which direction you are heading **toward**.

We now have a theorem:

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x) = L.$$

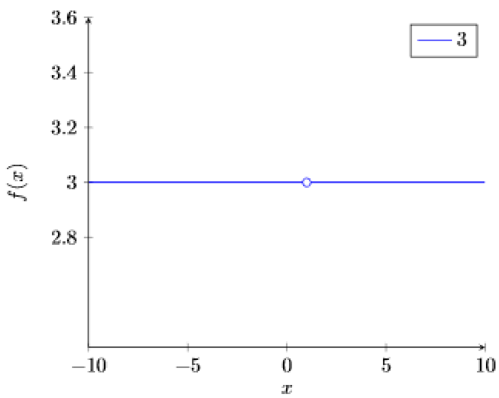
We see the left-handed and right-handed limits must be the same for the limit to exist. As we discussed in our rather ad hoc, non-formal example. (Note that ⇔ means if and only if: If the left hand side is true, then the right hand side is true and vice versa. We will use this notation in this text).

Let us consider this graph:



Try to evaluate $\lim_{x \to 0^-} f(x)$. We observe that the y-value is 2, when you approach 0 from the left. Therefore, $\lim_{x \to 0^-} f(x) = 2$. Let us now look at $\lim_{x \to 0^+} f(x)$. We see that the y-value is as close as you can get to 0 when you are as close to $x = 0$ from the right-side. Therefore, $\lim_{x \to 0^+} f(x) = 0$. We can now say that $\lim_{x \to 0} f(x)$ Does Not Exist. (We also observe that $f(0) = 2$. This fact has no bearing whatsoever on our limit, since a limit never reaches the function value on either side).

Now let's look at this graph:

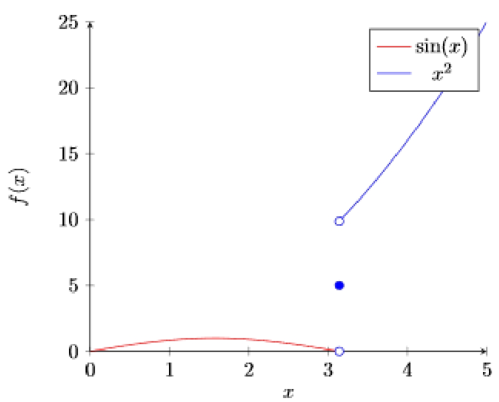


Let us calculate $\lim_{x \to 1} f(x)$

$\lim_{x \to 1^-} f(x) = 3 = \lim_{x \to 1^+} f(x)$. We conclude $\lim_{x \to 1} f(x) = 3$. We note that $f(1)$ does not exist and does not matter.

Now, take a look at $\lim_{x \to 5} f(x)$. Hopefully, you see that it exists, and equals 3.

EXAMPLE:



What is $\lim_{x \to \pi^-} f(x)$? It equals 0. And $\lim_{x \to \pi^+} f(x)$? It is 9. We also observe that $f(\pi) = 5$. What is the $\lim_{x \to \pi} f(x)$? It does not exist.

NUMERICAL LIMITS: We can also guess the value of a limit by substituting values closer and closer to $x = a$ into our calculator to see which value of L we are getting close to.

Example: $\lim_{x \to 2} \frac{x^2-4}{x-2}$:

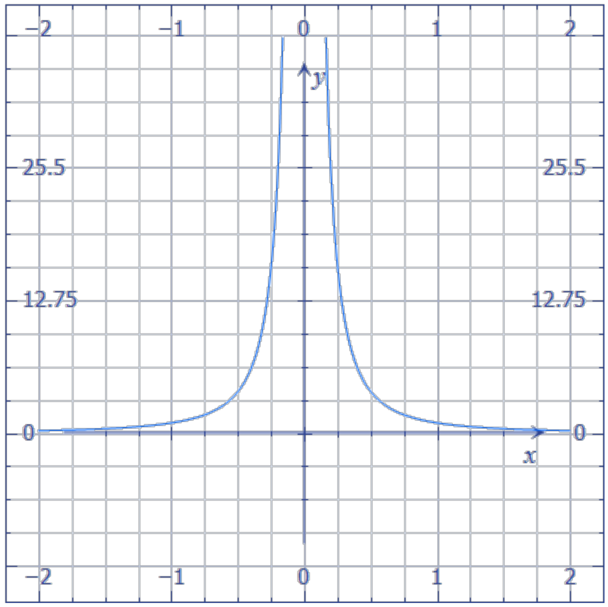
x	f(x)
1.9	3.9
1.99	3.99
1.999	3.999
2.1	4.1
2.01	4.01
2.001	4.001

What do you think the limit is? We guess 4. The closer we get to $x = 2$, the closer we get to $f(x) = 4$.

INFINITE LIMITS:

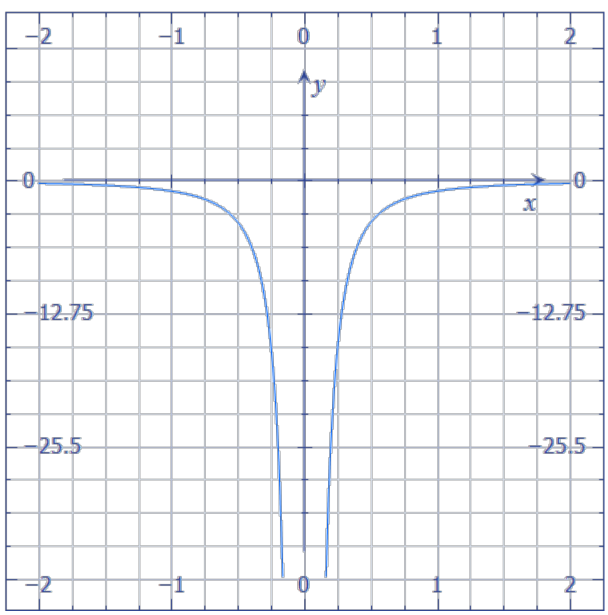
Let's start with an example: $f(x) = \frac{1}{x^2}$: We now consider the $\lim_{x \to 0} f(x)$. Let us take smaller and smaller values for x. As x becomes smaller, what happens to $f(x)$? We can plug smaller and smaller values of x into our calculator e.g. For each smaller value of x, we observe f becomes larger and larger. We also observe that whether x is either positive or negative, f remains positive. We know we cannot divide by 0, but we have also learned that a limit never reaches the value, so there is not a problem here with taking a limit as $x \to 0$. As x becomes smaller and smaller, in fact, being as arbitrarily close to 0 as possible, that f grows without bound. We will say that $\lim_{x \to 0} f(x) = \infty$.

The graph of this function is below:



So our intuitive definition of an infinite limit is $\lim_{x \to a} f(x) = \infty$, means the closer you get to $x \to a$ from either side, the larger $f(x)$ grows. This implies we can make $f(x)$ as large as we want by getting sufficiently close to a.

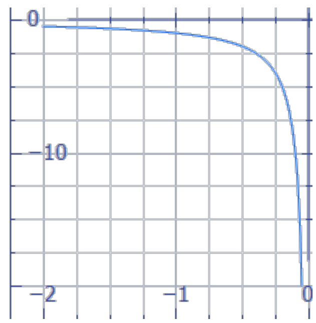
We can also take the case $\lim_{x \to a} f(x) = -\infty$, which means the closer you get to $x \to a$ from either side, the larger $f(x)$ grows negatively. E.g. $f(x) = -\frac{1}{x^2}$

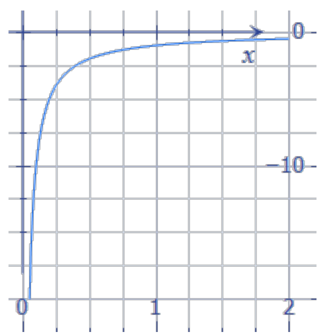


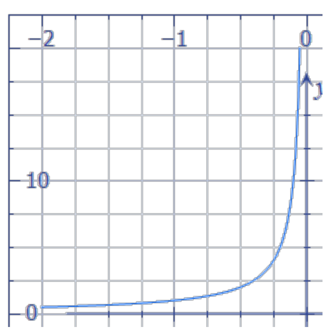
We can also consider one-sided infinite limits:

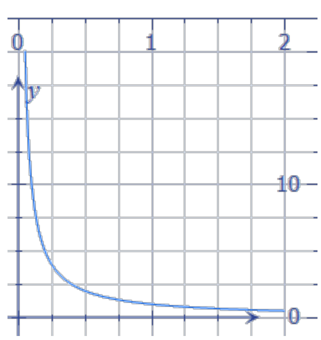
$\lim\limits_{x \to a^-} f(x) = -\infty$, $\lim\limits_{x \to a^+} f(x) = -\infty$, $\lim\limits_{x \to a^-} f(x) = \infty$, $\lim\limits_{x \to a^+} f(x) = \infty$.

An example of each:









DEFINITION: A vertical asymptote of $f(x)$ is the line $x = a$, whenever at least one of the following is true:

$\lim\limits_{x\to a^-} f(x) = -\infty$, $\lim\limits_{x\to a^+} f(x) = -\infty$, $\lim\limits_{x\to a^-} f(x) = \infty$, $\lim\limits_{x\to a^+} f(x) = \infty$, $\lim\limits_{x\to a} f(x) = \infty$, or $\lim\limits_{x\to a} f(x) = -\infty$

The above examples all had a vertical asymptote at $x = 0$.

EXAMPLE: Now let's consider an algebraic example of an infinite limit:

Consider the function $f(x) = \frac{1}{x-1}$:

Let's look at $\lim\limits_{x\to 1^-} f(x)$: We observe that the denominator is close to zero, so the function will blow up (increase without bound): We need to determine if it's going to $+\infty$, or $-\infty$. Since $x \to 1^-$, $f(x)$ will be negative, therefore $\lim\limits_{x\to 1^-} f(x) = -\infty$. Likewise $\lim\limits_{x\to 1^+} f(x) = \infty$.

EXAMPLE: Now, consider the function $f(x) = \frac{1}{\sin x} = \csc x$.

$\operatorname{Lim}\limits_{x\to 0^-} f(x) = -\infty$

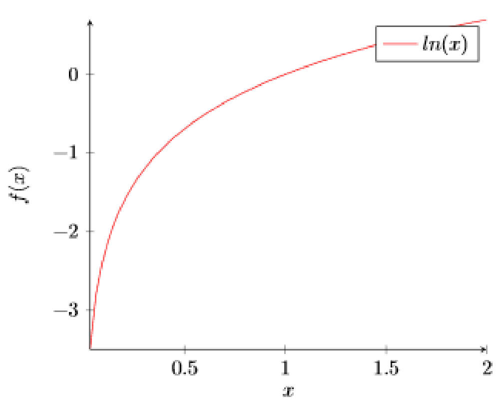
$\operatorname{Lim}\limits_{x\to 0^+} f(x) = \infty$

$\operatorname{Lim}\limits_{x\to \pi^-} f(x) = \infty$

$\operatorname{Lim}\limits_{x\to \pi^+} f(x) = -\infty$

Etc. (It has an infinite number of infinite limits).

EXAMPLE: We also have $f(x) = \ln(x)$:



Observe that $\lim_{x\to 0^+} f(x) = -\infty$, and $\lim_{x\to 0^-} f(x) = \text{DNE}$, as it is not in the domain of f. (DNE means Does Not Exist).

STEPS TO FIND A VERTICAL ASYMPTOTE:

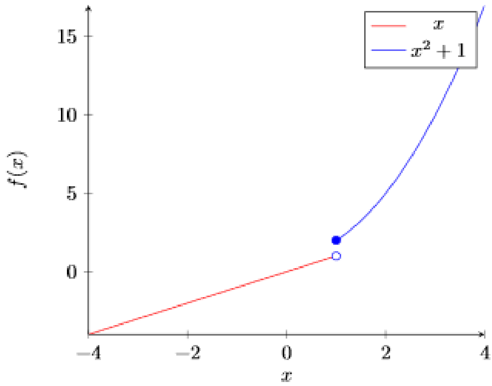
1) Set the denominator equal to zero, and solve for x.

   Case 1: There is no solution → there is no vertical asymptote. Example $\frac{1}{x^2+1}$. Since $x^2 + 1$ has no real solutions, there is no vertical asymptote.

   Case 2: You find a value or values for x, and it does not cancel a factor in the numerator. Those are vertical asymptote(s). Example: $\frac{1}{x^2-1}$ has two vertical asymptotes at $x = \pm 1$.

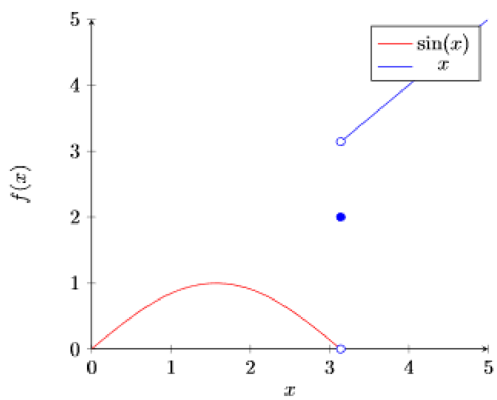
   Case 3: You have a solution, but it cancels a factor in the numerator. This is a hole and not a vertical asymptote. Example: $\frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)}$ has a hole at $x = -1$ and a vertical asymptote at $x = 1$.

EXERCISES

1) 

Let us consider the graph above:

a) Find $\lim_{x \to 1^-} f(x)$:
b) Find $\lim_{x \to 1^+} f(x)$:
c) Find $\lim_{x \to 1} f(x)$:
d) Find $\lim_{x \to 0} f(x)$:



2) Consider the graph above:

a) Find $\lim_{x \to \pi^-} f(x)$:
b) Find $\lim_{x \to \pi^+} f(x)$:

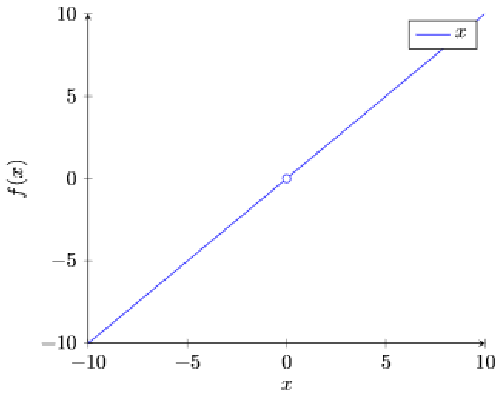
c) Find $f(\pi)$:

d) Find $\lim_{x \to \pi} f(x)$:

e) Find $\lim_{x \to 4} f(x)$:

f) Find $f(4)$:

3) Consider the graph:

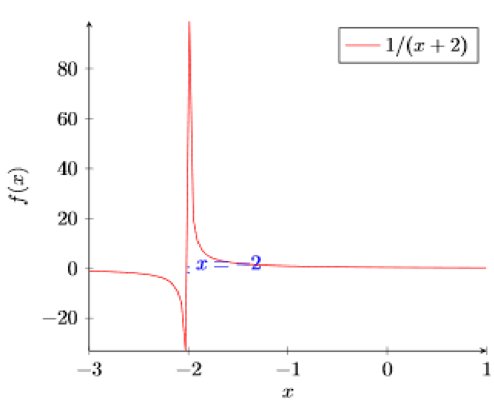


a) Find $\lim_{x \to 0^+} f(x)$:

b) Find $\lim_{x \to 0^-} f(x)$:

c) Find $\lim_{x \to 0} f(x)$:

4) Consider the graph:

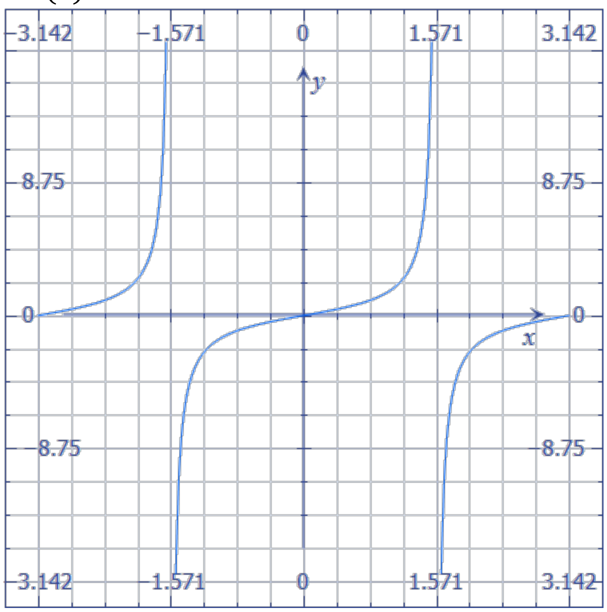


a) Find $\lim_{x \to -2^-} f(x)$:

b) Find $\lim_{x \to -2^+} f(x)$:

c) Find $\lim_{x \to -2} f(x)$:

5) $\tan(x)$



a) Find $\lim_{x \to \pi^-} f(x)$:

b) Find $\lim_{x \to \pi^+} f(x)$:

c) Find $\lim_{x \to \pi} f(x)$:

6) Guess the following limit:

$\lim_{x \to 3} \frac{x^2-9}{x-3}$:

x	f(x)
2.9	5.9
2.99	5.99
2.999	5.999
3.1	6.1
3.01	6.01
3.001	6.001

Find the following infinite limits:

7) $\lim_{x \to 1^+} \frac{3}{x^2-1}$:

8) $\lim_{x \to -2^+} \frac{2}{\sqrt{x+2}}$:

9) $\lim_{x \to 1^+} \frac{x+1}{x-1}$:

10) $\lim_{x \to 4^-} \frac{\sqrt{x}}{x-4}$:

11) $\lim_{x \to \frac{\pi}{2}^+} \tan(x)$:

12) $\lim_{x \to 1^+} \ln(x-1)$:

Find the following vertical asymptotes:

13) $f(x) = \frac{1}{x-7}$:

14) $f(x) = \frac{3}{x^2-5x+6}$

15) $f(x) = \frac{x-2}{x^2+2x-8}$

16) $f(x) = \frac{x+1}{x^3+1}$

17) $f(x) = tanx, 0 \leq x \leq 2\pi$

CHAPTER 1
SECTION 2

LIMITS USING LIMIT LAWS (ALGEBRAIC LIMITS)

In the previous section, we learned about limits, and took a graphical approach to our understanding. In this section, we will learn how to calculate limits given a function without a graph.

Let us first list all the limit laws:

Let c be a constant, and let $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then the following are true:

1) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2) $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$
3) $\lim_{x\to a}[f(x)\, g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
4) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
5) $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
6) $\lim_{x\to a} c = c$
7) $\lim_{x\to a} x = a$
8) $\lim_{x\to a} x^n = a^n$ where n is a positive integer
9) $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer, and if n is even, $a > 0$
10) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)}$ where n is a positive integer, and if n is even, $\lim_{x\to a} f(x) > 0$

Next, we will go over the steps to find a limit using limit laws:

1) Substitute x with a.

Case I: You get a finite number: You are done, and that is the limit.

Case 2: You get a finite number in the numerator, and approach zero in the denominator: You have an infinite limit. Use the techniques from the previous section to determine the limit. (Hint it will be either $+\infty$ or $-\infty$).

Case 3: You get a form of $\frac{0}{0}$. This is considered an indeterminate form. (Note: at first glance, you may think this case is undefined, since you know we cannot divide by zero. Recall that with a limit, you never reach the value, so this is indeterminate rather than undefined. The limit may or may not exist). When you get this form, you need to manipulate the function to get it out of this form, e.g. factoring and canceling. If $f(x) = g(x)$ when $x = a$, then

$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ if the limit exists. This allows us to simplify our limit, when $x = a$ gives us an indeterminate form. Since we never reach the value, we can simply cancel. We then recalculate the limit.

There can also be other indeterminate forms such as $\frac{\infty}{\infty}$, $\infty - \infty$, and more. We will deal with these individually.

Case 1 Example: Find $\lim_{x \to 2} x^2 - 2x + 4$: We substitute 2 for x and get 4.

Case 2 Example: Find $\lim_{x \to 0} \frac{1}{x^2}$: We substitute 0 for x and we get ∞, as we saw in the previous section.

Case 3 Example: Find $\lim_{x \to 2} \frac{x-2}{x^2-4}$. We get a form of $\frac{0}{0}$ here. We notice that the function is undefined at $x = 2$. This does not pose a problem for us, since we are evaluating a limit, and not evaluating $f(2)$. We can factor the bottom into $(x-2)(x+2)$, giving us $\lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$.

Example: Find $\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}$. This is another form of $\frac{0}{0}$. Again, we must factor. This becomes $\lim_{x \to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2} = \lim_{x \to 4} \sqrt{x} + 2 = 4$.

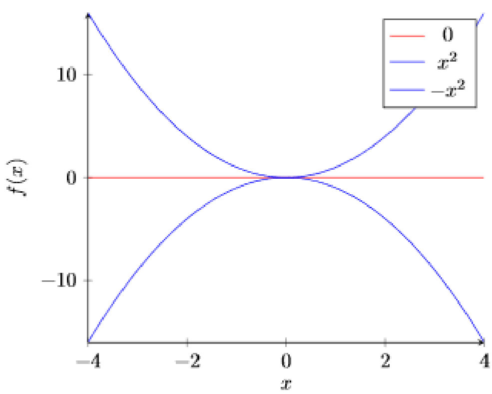
Example: Find $\lim_{t \to 0} \frac{\sqrt{t+1}-1}{t}$. Again we have form of $\frac{0}{0}$. In this case we will multiply numerator and denominator by the conjugate to simplify our expression. $\lim_{t \to 0} \frac{\sqrt{t+1}-1}{t} = \lim_{t \to 0} \frac{\sqrt{t+1}-1}{t} \cdot \frac{\sqrt{t+1}+1}{\sqrt{t+1}+1} = \lim_{t \to 0} \frac{t}{t(\sqrt{t+1}+1)} = \lim_{t \to 0} \frac{1}{\sqrt{t+1}+1} = \frac{1}{2}$.

Example: Let us now consider a limit that has the form of , $\infty - \infty$. This is another indeterminate form that will be new to us. $\lim_{x \to 0^+} \frac{1}{x^2+x} - \frac{1}{x}$ is of this form. The first thing that comes to mind is to find a common denominator: We get $\lim_{x \to 0^+} \frac{-x}{x^2+x} = \lim_{x \to 0^+} -\frac{1}{x+1} = -1$.

Example: Let us consider $\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$: We observe that this is where the sine function is approaching ∞. What does this mean? We know that the sine function oscillates indefinitely between -1 and 1, so it never settles on any value as you approach $\pm\infty$. We determine this limit does not exist.

Theorem: If $f(x) \leq g(x)$ when x approaches a (except possibly at a), and both limits exist , then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ when x approaches a (except possibly at a), and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.



You should be able to see from the above graph, that all three functions have the same limit (and in this case, the same function value) at $x = 0$.

Example: If $2 \leq f(x) \leq x^2 + 2$ for all x, find $\lim_{x \to 0} f(x)$. We use the Squeeze Theorem. We observe $f(x)$ meets the first condition for the Squeeze Theorem. Next, we calculate the limits on the LHS, and the RHS of f. The $\lim_{x \to 0} 2 = 2$, and, $\lim_{x \to 0} x^2 + 2 = 2$. Therefore, $\lim_{x \to 0} f(x) = 2$ by the Squeeze Theorem.

Example: Let us now look at $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$. We observed in a previous example that the $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist. But with this example, we can use the Squeeze Theorem. We observe that

$0 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for all x. Next, we multiply all sides by x. We now have $0 \leq x \sin\left(\frac{1}{x}\right) \leq x$. We take $\lim_{x \to 0} 0 = 0$. And $\lim_{x \to 0} x = 0$. Therefore $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$ by the Squeeze Theorem.

Exercises:

Find the following limits (if they exist):

1) $\lim_{x \to 1} x^2 + 2x - 5$:

2) $\lim_{x \to 0} \frac{1}{2x^4}$:

3) $\lim_{x \to -1} \frac{x^2-1}{x+1}$:

4) $\lim_{x \to 2} \frac{x^2-5x+6}{x-2}$:

5) $\lim_{x \to -2} \frac{2x^2+3x-2}{x+2}$:

6) $\lim_{x \to 3} \frac{3x^2-10x+3}{x-3}$:

7) $\lim_{x \to 1} \frac{x^2-1}{x^4-1}$:

8) $\lim_{x \to 1} \frac{x^3-1}{x-1}$:

9) $\lim_{x \to 1} \frac{x^3-2x^2+x}{x-1}$:

10) $\lim_{x \to 0} \frac{x^4+5x^2-2x}{x}$:

11) $\lim_{x \to 0^+} \frac{\sqrt{x+5}-\sqrt{5}}{x}$:

12) $\lim_{x \to 9} \frac{\sqrt{x}-3}{9-x}$:

13) $\lim_{t \to 0^-} \frac{3-\sqrt{9-t}}{t}$:

14) $\lim_{h \to 0} \frac{(3+h)^2-9}{h}$:

15) $\lim_{t \to 0} \frac{(x+t)^2-x^2}{t}$:

16) $\lim_{t \to 0} \frac{1}{t} - \frac{1}{t^3+t}$:

17) Use the Squeeze Theorem to find the following limits:

a) Let $x^2 + 1 \leq f(x) \leq e^x$ for $x \geq 0$ Find $\lim_{x \to 0} f(x)$:

b) Find $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$:

CHAPTER 1
SECTION 3

PRECISE DEFINITION OF A LIMIT

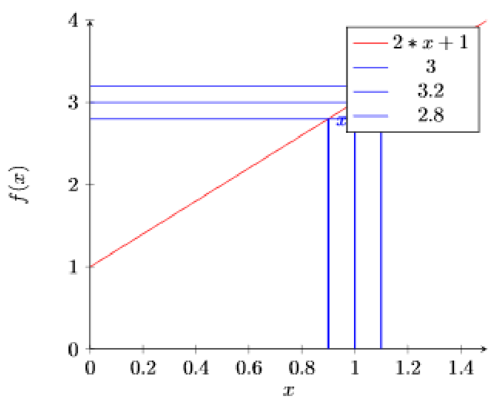
Let us discuss what a "precise" definition of a limit is, and why we would need such a thing. We already have the definition: $\lim_{x \to a} f(x) = L$ as our intuitive definition. So what is wrong with this definition? It has seemed to have served us well enough thus far. The problem is in the vague concepts of x "approaching" a, and being "arbitrarily" close to L. What do those terms actually mean? We need something more precise. E.g. exactly how close to a do we need to be, in order to be a specified distance from L?

Let's consider the $\lim_{x \to 1}(2x + 1) = 3$:

Previously, we would say that as x approaches 1 from either side, we get arbitrarily close to 3. So the limit is 3.

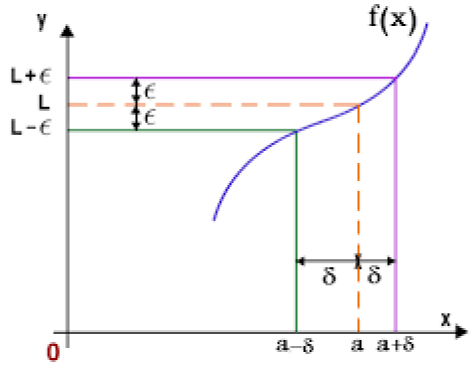
But how precise is this really? What if we, say want to be within .2 of 3 on either side? How close to 1 does x actually need to be?

We will discover with our precise definition that it needs to be within .1 of x =1.



We will prove this using the formal definition (forthcoming), but the above graph gives us a graphical interpretation, which we can visually confirm. (Proof by picture!)

Let us now take a look at a slightly more abstract graph:



We observe the same kind of situation as the example we just discussed, except this time there are no values. We see that the distance away from L is now ϵ, and the distance away from $x = a$ is δ. We also observe from this graph (which is non-linear), that in order to get as close as a distance ϵ from L, we must take the smaller distance for δ, or both sides of L will not be within ϵ.

We are now ready to write down our formal definition:

PRECISE DEFINITION OF A LIMIT: Let f be a function defined on an open interval containing a, except possibly at a itself. Then $\lim_{x \to a} f(x) = L$ means if for every number $\epsilon > 0$, there exists a number $\delta > 0$, such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Let's talk about what this means. It means that if the limit exists and is L, then there exists some value δ (in the x-direction), that allows us to be as close to L as we need to be, i.e. within an ϵ away. We note that $|x - a|$ is the distance between x and a, and $|f(x) - L|$ is the distance between $f(x)$ and L. (Note: I am going to point out the obvious, because students get confused: most notably that $|f(x) - L|$ is in the y-direction).

We will be doing general proofs using this definition. Before doing that, let us return to our **previous example**: $\lim_{x \to 1}(2\,x + 1) = 3$:

The first thing you may notice, is that we already need to have calculated the limit (using techniques from section 1.1) before we can use any of these new techniques. We want to show that in order to be within a .2 distance from 3, we must be within a .1 distance from $x = 1$. We will use our new definition.

We first start with $0 < |x - a| < \delta$. This is the "if", which means it is the given. We always assume this part is true before proceeding further. In our case, we have $0 < |x - 1| < \delta$. We will next look at the fact that we want $|f(x) - L| < .2$. Let us manipulate $|f(x) - L|$. We have $f(x) = 2x + 1$. We have $L = 3$. So $|f(x) - L| < .2 \rightarrow |(2x + 1) - 3| < .2$. This implies $|2x - 2| < .2 \rightarrow 2|x - 1| < .2$. Since $0 < |x - 1| < \delta$, then it is easy to see that δ must equal .1

Let us now use this definition to find a general case for our example. We will find a δ in terms of ϵ, that will work for any $\epsilon > 0$.

We do this in two parts: First we do what we call **preliminary work**, in order to find our δ in terms of ϵ. After finding our δ, we will construct a formal proof.

Let f be a function defined on an open interval containing a, except possibly at a itself. Then $\lim_{x \to a} f(x) = L$ means if for every number $\epsilon > 0$, there exists a number $\delta > 0$, such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Let us rewrite what we want to prove for our particular example. **:** We want to prove that $\lim_{x \to 1} |(2x + 1) - 3| < \epsilon$ means if for every number $\epsilon > 0$, there exists a number $\delta > 0$, such that if $0 < |x - 1| < \delta$, then $|(2x + 1) - 3| < \epsilon$.

PRELIMINARY WORK: We need to find a δ that will work for every $\epsilon > 0$. We start by manipulating

$|(2x + 1) - 3| = |2x - 2| = 2|x - 1|$. Next we go back to what we already know. Recall that

$0 < |x - 1| < \delta$ is the if (or the given). This is the part we have to assume is true. So if $0 < |x - 1| < \delta$ is true, then $0 < 2|x - 1| < 2\delta$ (by using basic algebra). But we want $2|x - 1| < \epsilon$. We observe that if $\delta < \frac{\epsilon}{2}$, then this will work. We already have $0 < 2|x - 1| < 2\delta$. So if $\delta = \frac{\epsilon}{2}$, then

$0 < 2|x - 1| < 2\left(\frac{\epsilon}{2}\right) \rightarrow\ 0 < 2|x - 1| < \epsilon$. So our choice worked.

Next, we rewrite this as a formal proof, basically going backwards.

FORMAL PROOF: For every $\epsilon > 0$, there exists a $\delta > 0$ such that If $0 < |x - 1| < \delta$, then $0 < 2|x - 1| < 2\delta \rightarrow\ |(2x + 1) - 3| < 2\delta$. But we let $\delta = \frac{\epsilon}{2}$. Therefore $|(2x + 1) - 3| < \frac{2\epsilon}{2} \rightarrow$

$|(2x + 1) - 3| < \epsilon$. Q.E.D.

(**Q.E.D.** is an abbreviation of the Latin words "Quod Erat Demonstrandum" which loosely translated means "that which was to be demonstrated". It is usually placed at the end of a mathematical proof to indicate that the proof is complete.)

<u>EXAMPLE</u>: Let's try another similar one: Prove $\lim_{x \to 2}(3\,x - 2) = 4$:

PRELIMINARY WORK: We start by manipulating $|(3x - 2) - 4|$ = $|3x - 6| = 3|x - 2|$. Recall that $0 < |x - 2| < \delta$ is the if (or the given). This is the part we have to assume is true. So if $0 < |x - 2| < \delta$ is true, then $0 < 3|x - 2| < 3\delta$. We want $3|x - 2| < \epsilon$. We choose $\delta = \frac{\epsilon}{3}$. We have $0 < 3|x - 2| < 3\delta$ $\rightarrow 0 < 3|x - 2| < 3\left(\frac{\epsilon}{3}\right) \rightarrow\ 0 < 3|x - 2| < \epsilon$.

FORMAL PROOF: For every $\epsilon > 0$, there exists a $\delta > 0$ such that If $0 < |x-2| < \delta$, then $0 < 3|x-2| < 3\delta \rightarrow |(3x-2)-4| < 3\delta$. But we let $\delta = \frac{\epsilon}{3}$. Therefore-$|(3x-2)-4| < \frac{3\epsilon}{3} \rightarrow |(3x-2)-4| < \epsilon$. Q.E.D.

EXAMPLE: Let's do one more linear example, this time with fractions. Prove $\lim_{x\to 2}(2-\frac{1}{2}x) = 1$:

PRELIMINARY WORK: We start with $\left|(2-\frac{1}{2}x)-1:\right|$ = $\left|-\frac{1}{2}x+1\right| = \left|\frac{1}{2}x-1\right|$ (because of the absolute value). Then, it equals $\frac{1}{2}|x-2|$. Recall that $0 < |x-2| < \delta$ is the if (or the given). So if $0 < |x-2| < \delta$ is true, then $\frac{1}{2}|x-2| < \frac{1}{2}\delta$. We want $\frac{1}{2}|x-2| < \epsilon$. We choose $\delta = 2\epsilon$. We have $0 < \frac{1}{2}|x-2| < \frac{1}{2}\delta \rightarrow 0 < \frac{1}{2}|x-2| < \frac{1}{2}(2\epsilon) \rightarrow 0 < \frac{1}{2}|x-2| < \epsilon$.

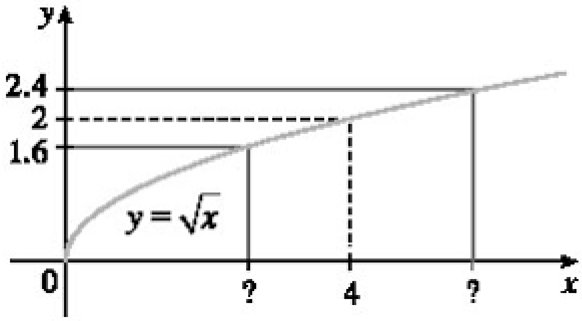
FORMAL PROOF: For every $\epsilon > 0$, there exists a $\delta > 0$ such that If $0 < |x-2| < \delta$, then $0 < \frac{1}{2}|x-2| < \frac{1}{2}\delta \rightarrow \left|\left(\frac{1}{2}x-1\right)\right| < \frac{1}{2}\delta \rightarrow |(2-\frac{1}{2}x)-1| < \frac{1}{2}\delta$:. But we let $\delta = 2\epsilon$. Therefore: $\left|(2-\frac{1}{2}x)-1\right| < \frac{2\epsilon}{3} \rightarrow \left|(2-\frac{1}{2}x)-1\right| < \epsilon$. Q.E.D.

EXAMPLE: Next we will try something a little harder. We are going to try a nonlinear example. Prove $\lim_{x\to 1} x^2 = 1$.

PRELIMINARY WORK: We have $|x^2-1| = |x+1||x-1|$. We know $0 < |x-1| < \delta$. Then $|x+1||x-1| < |x+1|\delta$. We need to find a positive constant C, such that $|x+1| < C$. Then $|x+1||x-1| < C\delta$. Let $\delta = \frac{\epsilon}{C}$. We need to find C. Recall, since we are proving a limit, it means that we are close to $x=1$. Therefore, let us assume we are within one value away from $x=1$. Therefore $|x-1| < 1$. So $0 < x < 2 \rightarrow 1 < x+1 < 3$. And, $|x+1| < 3$. So we can now choose C to be 3. So $|x-1| < \frac{\epsilon}{3}$. But we also had the previous inequality $|x-1| < 1$. Recall that δ must be the smallest value to ensure that we will be within an ϵ distance away from our limit L. Therefore δ must be the minimum of $\frac{\epsilon}{3}$ and 1.

FORMAL PROOF: For every $\epsilon > 0$, there exists a $\delta > 0$ such that If $0 < |x-1| < \delta$, then$|x-1| < 1 \rightarrow 0 < x < 2 \rightarrow 1 < x+1 < 3 \rightarrow |x+1| < 3$. We also have $|x-1| < \frac{\epsilon}{3}$ so $|x^2-1|$ = $|x+1||x-1| < C\delta \rightarrow |x+1||x-1| < 3\delta$. We let $\delta = \min\{1, \frac{\epsilon}{3}\} \rightarrow |x+1||x-1| < \frac{3\epsilon}{3} = \epsilon$, Q.E.D.

EXERCISES:

1) 

Find the values for δ given the values of ϵ.

Find the values of δ, for the following function if $\epsilon = .1$ on both sides:

2) $\lim_{x\to 2}(5x - 1) = 9$:

3) $\lim_{x\to 2}(\frac{1}{2}x + 1) = 2$:

Prove the following limits using our Precise Definition of a limit:

4) $\lim_{x\to 1}(4x - 3) = 1$:

5) $\lim_{x\to 2}(6x - 6) = 6$:

6) $\lim_{x\to 5}(x - 1) = 4$:

7) $\lim_{x\to 3}(5x + 1) = 16$:

8) $\lim_{x\to 2}(\frac{1}{2}x + 1) = 2$:

9) $\lim_{x\to \frac{1}{5}}(5x + 2) = 3$:

10) $\lim_{x\to 3}(2 - \frac{1}{3}x) = 1$:

11) $\lim_{x \to 2} \mathrm{c} = c$:

12) $\lim_{x \to 1} \mathrm{x} = 1$:

13) $\lim_{x \to 1} 2\mathrm{x}^2 = 2$:

14) $\lim_{x \to 2} x^2 = 4$:

CHAPTER 1
SECTION 4

CONTINUITY

Before we define continuity, let us first discuss a general idea of what it might mean. An idea of continuity can be: If I have to pick up my pencil at any point while drawing a graph, it would not be continuous there. (**NOTE: THIS IS NOT A DEFINITION OF CONTINUITY, FORMAL OR OTHERWISE).** It is simply a concept, not well-defined, merely to get your brain thinking about it.

Let us now formally define continuity: A function f is continuous at a value a, if $\boldsymbol{\lim_{x \to a} f(x) = f(a)}$.

This definition implies 3 things:

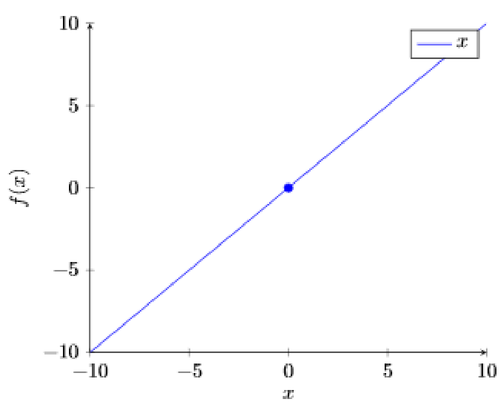
1) $\lim_{x \to a} f(x)$ exists.

2) $f(a)$ exists (the function is defined at a).

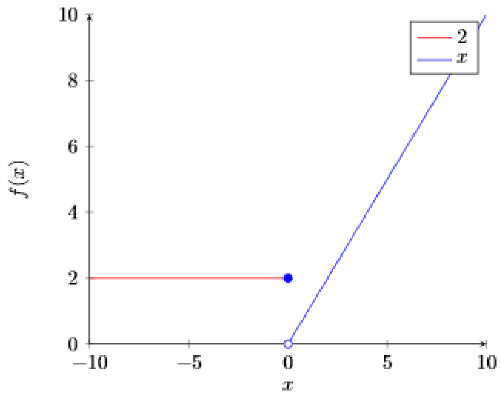
3) $\lim_{x \to a} f(x) = f(a)$ implies that they must also equal each other.

If a function f is continuous at $x = a$, then one needs only state the definition. If it is not continuous, one will list the condition that failed.

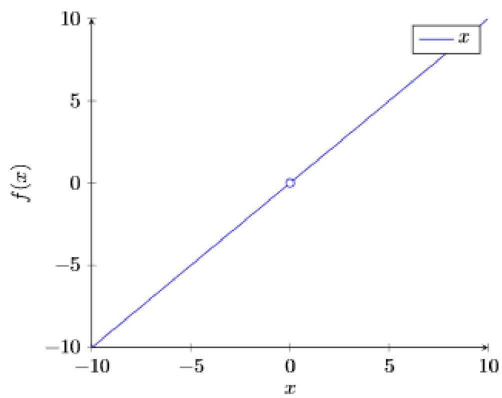
Next, we will demonstrate graphically, 4 different cases: One, which is continuous at $x = 0$, and 3 that are not, each for a different reason:



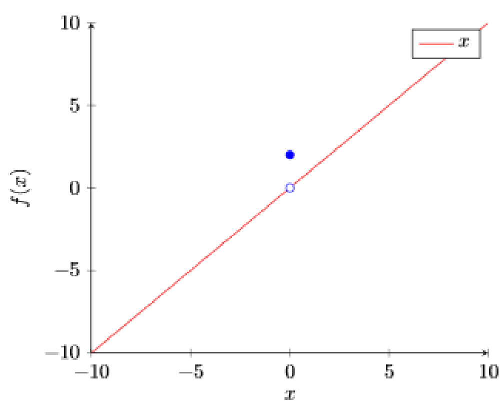
We quickly observe $\lim_{x \to 0} f(x) = f(0) = 0$. So we conclude $f(x)$ is continuous at $x = 0$.



Here, we observe $f(x)$ is not continuous at $x = 0$. Can you determine which condition failed? Hopefully you can see that $\lim_{x\to 0} f(x)$ does not exist. $\lim_{x\to 0^-} f(x) = 2$, and $\lim_{x\to 0^+} f(x) = 0$.

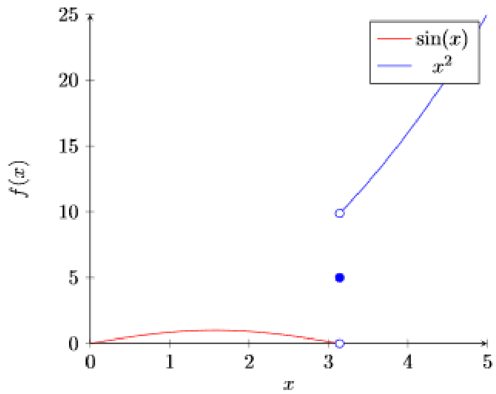


In this graph, we also observe $f(x)$ is not continuous at $x = 0$. Why not? We see that $f(0)$ does not exist here, (or f is not defined at $x = 0$).



This graph also is not continuous at $x = 0$. We observe $\lim_{x \to 0} f(x) = 0$, so it does exist. We also see $f(0)$ exists, and it is 2. However, these 2 values are not equal, therefore $f(x)$ is not continuous at $x = 0$.

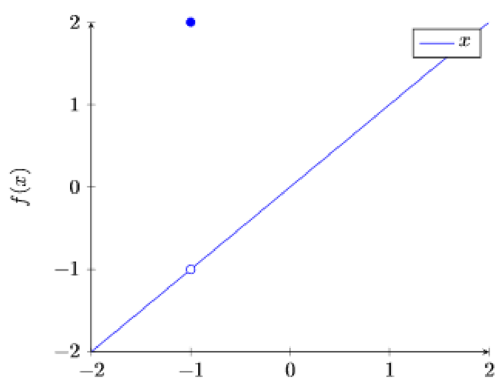
EXAMPLE: Let us go back to a graph from Section 1.1:



Is $f(x)$ continuous at $x = \pi$? Why or why not? We observe that it is not. Why? $\lim_{x \to \pi} f(x)$ does not exist. What about at $x = 4$? Yes, $\lim_{x \to 4} f(x) = f(4) = 16$.

EXAMPLE: An example of a function that is discontinuous at $x = 0$: $f(x) = \frac{1}{x}$: We quickly observe that $f(0)$ does not exist. (We did not need a graph here).

EXAMPLE: Let $f(x) = \begin{cases} 2 \text{ when } x = -1 \\ x \text{ when } x \neq -1 \end{cases}$ We observe that $\lim_{x \to -1} f(x) = -1 \neq f(-1) = 2$. Therefore f is not continuous at $x = -1$.



THEOREM: If f, g are both continuous at $x = a$, and c is a constant, then the following functions are also continuous at $x = a$:

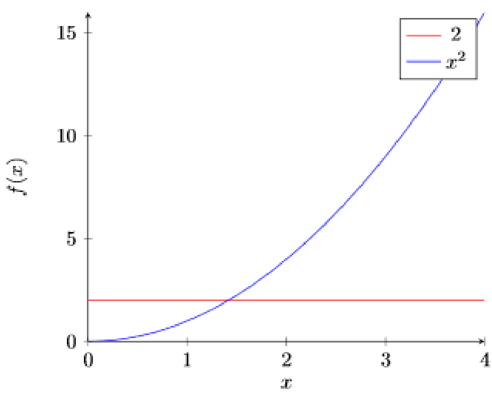
1) $f \pm g$
2) cf
3) fg
4) $\frac{f}{g}$ where $g(a) \neq 0$.

(The proofs are left as an exercise).

THEOREM: If g is continuous at $x = a$ and f is continuous at $g(x)$, then $f°g(x) = f(g(x))$ is also continuous at $x = a$.

THE INTERMEDIATE VALUE THEOREM: Let f be continuous on the closed interval $[a, b]$, and let N be a number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a value $x = c$, in (a, b) Note: It is now the open interval), such that $f(c) = N$.

This theorem is extremely useful to show there is a root in some interval of a continuous function. It may be difficult to find the root, but with this theorem, we can narrow it down, and use other methods. E.g, if a is negative, and b is positive, and f is continuous, we know there is a root in between the two values.



Notice this graph has a value between $x = 1$ and $x = 3$: The value $f(x) = 2$ is one such value as this graph shows. (We note that it is more difficult to find the x-value to which it corresponds).

EXAMPLE: Show there is a root of the equation: $x^3 + 2x^2 - 7$ between 1 and 2.

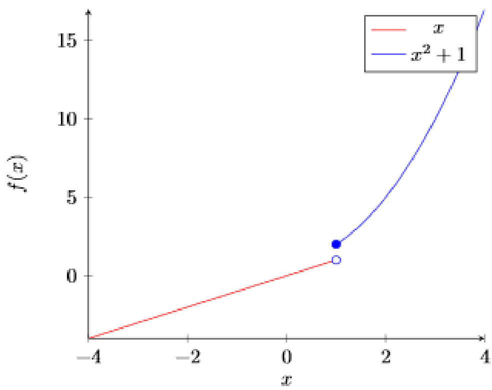
Let $f(x) =: x^3 + 2x^2 - 7$. f is a polynomial, which is continuous everywhere (proof left to the reader). Therefore f is continuous on $[1,2]$. Also $f(1) \neq f(2)$. Therefore, the Intermediate Value Theorem applies. This means there exists a value $x = c$ such that c is in (1,2) such that $f(c) = N$. Since $f(1) = -4$, which is negative, and $f(2) = 9$ which is positive, we conclude there is a value between 1 and 2 such that $f(c) = 0$. Therefore, there is at least one root in the interval (0,1).

A **removable** discontinuity is a hole in the function: E.g. $f(x) = \frac{x^2-1}{x-1}$ has a hole at $x = 1$, (as discussed in section 1.1), therefore it is a removable discontinuity. (We note that a vertical asymptote is not a removable discontinuity).

DEFINITION: A function f is continuous from the right at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

And a function f is continuous from the left at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

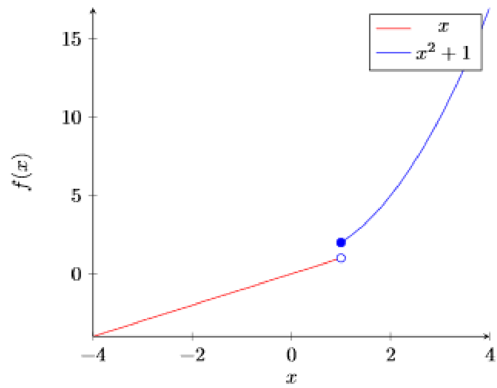
EXAMPLE:



$f(x)$ is continuous from the right but not from the left.

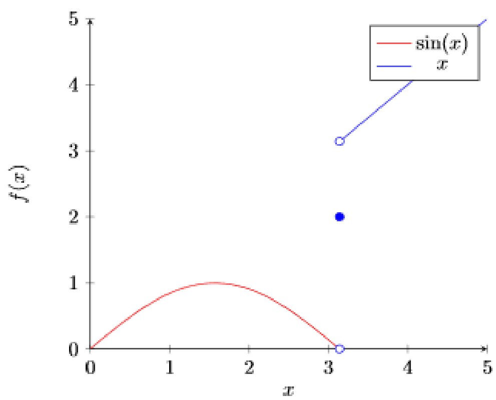
EXERCISES:

1) Refer to the graph below (from Exercise 1) of Section 1.1):



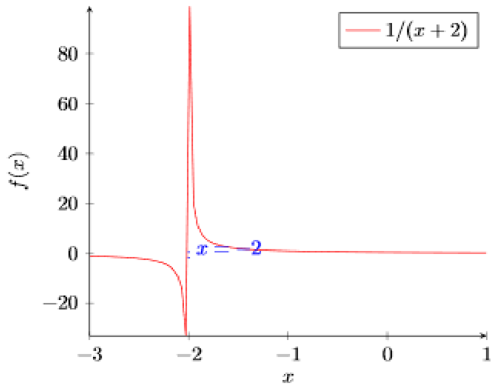
a) Is f continuous at $x = 1$? Why or why not?
b) Is f continuous at $x = 3$? Why or why not?

2) Refer to the graph below (from Exercise 2) of Section 1.1):



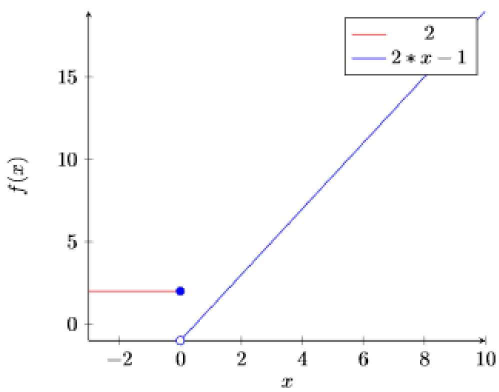
a) Is f continuous at $x = \pi$? Why or why not?
b) Is f continuous at $x = 2$? Why or why not?

3) Refer to the graph below (from Exercise 3) of Section 1.1):



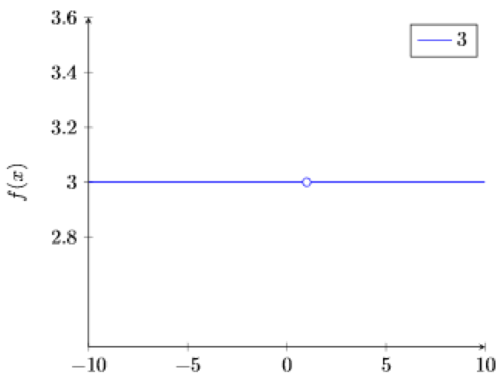
a) Is f continuous at $x = -2$? Why or why not?
b) Is f continuous at $x = 0$? Why or why not?

4) The graph below is an example from Section 1.1:



a) Is f continuous at $x = -2$? Why or why not?
b) Is f continuous at $x = 0$? Why or why not?
c) Is there a left or right continuity? If so, which and where?

5)



a) Is f continuous at $x = 0$? Why or why not?

b) Is there a removable discontinuity? If so, where?

Explain why f is discontinuous at the given value for $x = a$. Sketch the graph.

6) $f(x) = \frac{1}{x-4}$ $a = 4$:

7) $f(x) = \begin{cases} -x, & x < 1 \\ 3x, & x \geq 1 \end{cases}$ $a = 1$

8) $f(x) = \begin{cases} x+1, & x < -1 \\ 2x-1, & x \geq -1 \end{cases}$ $a = -1$

9) $f(x) = \begin{cases} x^2, & x < \pi \\ \sin x, & x \geq \pi \end{cases}$ $a = \pi$

10) $f(x) = \begin{cases} -x+1, & x < 0 \\ 0, & x = 0 \\ \frac{1}{x}, & x > 0 \end{cases}$ $a = 0$

Use the Intermediate Value Theorem to show a root exists for the equation in the given interval:

11) $x^3 - x + 1 = 0$, $(-2,0)$:

12) $\sin x = x$, $(-\pi, \pi)$:

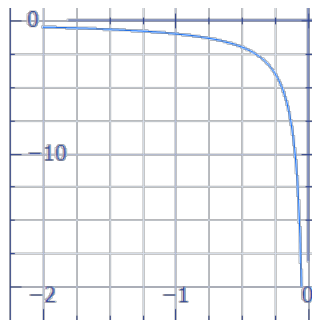
13) $x^5 - 2x^2 = 1$, (0,2):

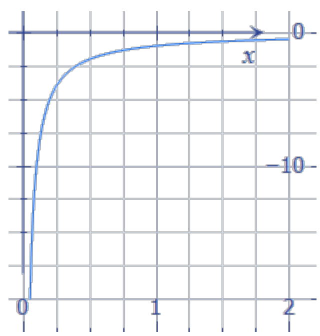
14) $e^x = 1$, (-1,1):

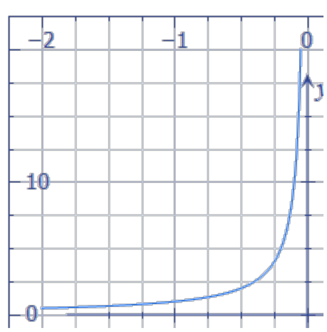
CHAPTER 1
SECTION 5

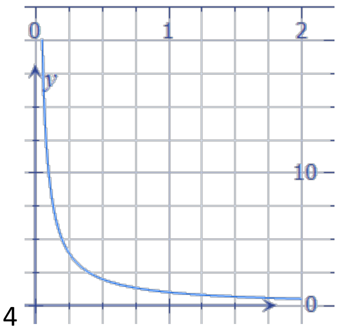
LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

Below are graphs from Section 1.1: We used these to learn about infinite limits and vertical asymptotes. We observe from these same graphs that each one seems to have a limit at infinity that coincide with $f(x) \to 0$. All these, in fact have a horizontal asymptote at $y = 0, or\ f(x) = 0$, and have limits at infinity that are 0.

1 

2 

3 



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DEFINITION: LIMIT AT INFINITY: (INTUITIVE): Let f be a function defined on (a, ∞), then $\lim_{x\to\infty} f(x) = L$ means that $f(x)$ can be made arbitrarily close to L by making x as large as is necessary.

Also: **LIMIT AT $-\infty$: (INTUITIVE):** Let f be a function defined on $(-\infty, a)$, then $\lim_{x\to-\infty} f(x) = L$ means that $f(x)$ can be made arbitrarily close to L by making x as large as is necessary in the negative direction.

NOTE: Above graphs: 2 and 4 are the limits at positive infinity where $\lim_{x\to\infty} f(x) = 0$, and graphs 1 and 3 are the limits at negative infinity: $\lim_{x\to-\infty} f(x) = 0$. They all have horizontal asymptotes at $y = 0$.

<u>DEFINITION:</u> The line $y = L \; or \; f(x) = L$ is a horizontal asymptote of the curve $y = f(x)$ if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ are true, or both are true.

THEOREM: Let $r > 0$ as a rational number, then $\lim_{x\to\pm\infty} \frac{1}{x^r} = 0$.

Without a formal proof, this is intuitively obvious. If you don't immediately see it, try dividing the number one by larger and larger values, and observe the number becomes smaller and smaller. The larger the denominator becomes, the smaller the number is. Since our denominator is x^r, and $r > 0$, the denominator is getting very large in absolute value at $\pm\infty$. Also, note that if you have any finite number on top, this Theorem also holds. By limit laws: (Limit Law 2): $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$. Therefore, for any number c, $\lim_{x\to\pm\infty} \frac{c}{x^r} = c \lim_{x\to\pm\infty} \frac{1}{x^r} = c(0) = 0$. (This Theorem will be most helpful in calculating limits at infinity and horizontal asymptotes).

STEPS TO FIND A HORIZONTAL ASYMPTOTE:

1) Take $\lim_{x\to\infty} f(x), \; or \; \lim_{x\to-\infty} f(x), \; or \; \lim_{x\to\pm\infty} f(x)$

a) If you have a form of $\frac{c}{x^r}$, the limit is 0. (In fact, If you have a constant c in the numerator and multiple terms of x^r, in the denominator, this still holds). **Example:** $f(x) = \frac{5}{x^2-5x+7}$. $\lim_{x\to\pm\infty} f(x) = 0$.

b) If you have a form of $\frac{\infty}{\infty}$, this is an indeterminate form. We must get the limit out of this form. The strategy is to divide every term in both the numerator and denominator by the highest power of x in the denominator, then simplify, and try the limit again. **Example:** $f(x) = \frac{x^2-5x-2}{3x^2-7}$. We attempt to calculate $\lim_{x\to\infty} \frac{x^2-5x-2}{3x^2-7}$. We quickly see this is a form of $\frac{\infty}{\infty}$. We now divide all terms by the highest power of x in the denominator which is x^2. Therefore: $\lim_{x\to\infty} \frac{x^2-5x-2}{3x^2-7} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}-\frac{5x}{x^2}-\frac{2}{x^2}}{\frac{3x^2}{x^2}-\frac{7}{x^2}} = \lim_{x\to\infty} \frac{1-\frac{5}{x}-\frac{2}{x^2}}{3-\frac{7}{x^2}}$. We see by our Theorem that all terms but 1 and 3 go to zero. Therefore $\lim_{x\to\infty} f(x) = \frac{1}{3}$, and the horizontal asymptote is $y = \frac{1}{3}$.

EXAMPLE: Find both vertical and horizontal asymptotes for $f(x) = \frac{4x^2+3x+1}{x^2-4}$. To find the vertical asymptote(s): Set $x^2 - 4 = 0 \rightarrow x = \pm 2$. Since they cancel no factors in the numerator, these are the vertical asymptotes. To find the horizontal asymptote, we take $\lim_{x\to\infty} \frac{4x^2+3x+1}{x^2-4}$. We get a form of $\frac{\infty}{\infty}$. We divide all terms by x^2. We get $\lim_{x\to\infty} \frac{\frac{4x^2}{x^2}+\frac{3x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{4}{x^2}} = \lim_{x\to\infty} \frac{4+\frac{3}{x}+\frac{1}{x^2}}{1-\frac{4}{x^2}} = 4$. So the horizontal asymptote is $y = 4$.

EXAMPLE: Find the vertical and horizontal asymptotes for $f(x) = \frac{x^3+x-1}{x^4-81}$. Setting $x^4 - 81 = 0$ gives $x = \pm 3$ for our vertical asymptotes. For our horizontal asymptote: $\lim_{x\to\infty} \frac{x^3+x-1}{x^4-81} = \lim_{x\to\infty} \frac{\frac{x^3}{x^4}+\frac{x}{x^4}-\frac{1}{x^4}}{\frac{x^4}{x^4}-\frac{81}{x^4}} =$ $\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^3}-\frac{1}{x^4}}{1-\frac{81}{x^4}}$. Notice here that all the terms go to 0, except for 1 in the denominator. So $\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^3}-\frac{1}{x^4}}{1-\frac{81}{x^4}} = \frac{0}{1} = 0$. So our horizontal asymptote is $y = 0$.

EXAMPLE: Find the vertical and horizontal asymptotes for $f(x) = \frac{x^3+2x+9}{x^2+8x}$. Setting the denominator equal to 0 gives us $x = 0, -8$ for the vertical asymptotes. For the horizontal asymptote: We take $\lim_{x\to\infty} \frac{x^3+2x+9}{x^2+8x} = \lim_{x\to\infty} \frac{\frac{x^3}{x^2}+\frac{2x}{x^2}+\frac{9}{x^2}}{\frac{x^2}{x^2}+\frac{8x}{x^2}} = \lim_{x\to\infty} \frac{x+\frac{2}{x}+\frac{9}{x^2}}{1+\frac{8}{x}} = \lim_{x\to\infty} \frac{x}{1} = \infty$. Therefore, there is no horizontal asymptote. Also $\lim_{x\to-\infty} \frac{x}{1} = -\infty$. In this particular case, there would be a slant asymptote, which we will cover later. These occur when the degree of the numerator is one degree greater than the degree of the denominator. (You find them by performing long division).

EXAMPLE: Find the vertical and horizontal asymptotes for $f(x) = \frac{\sqrt{x^2+5x-3}}{x+4}$. The vertical asymptote is $x = -4$. Horizontal: $\lim_{x\to\infty} \frac{\sqrt{x^2+5x-3}}{x+4}$.

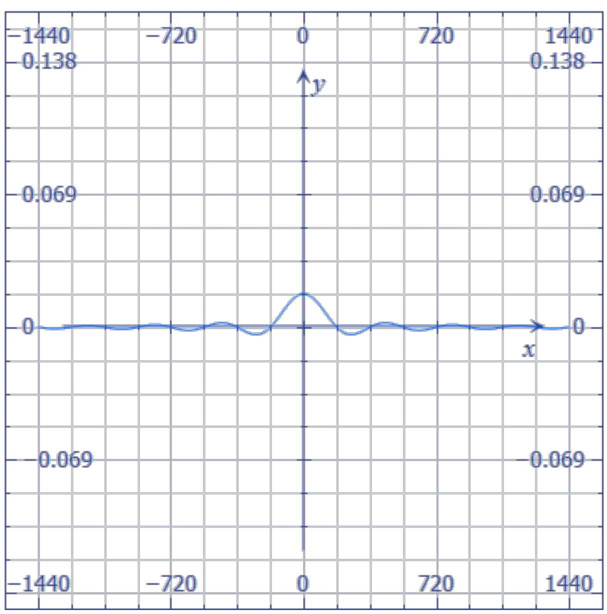
We still divide all terms by the highest power of x in the denominator, which in this case is x. :

$\lim_{x\to\infty} \frac{\sqrt{x^2+5x-3}}{x+4} =: \lim_{x\to\infty} \frac{\frac{\sqrt{x^2+5x-3}}{x}}{\frac{x+4}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{x^2+5x-3}{x^2}}}{\left(1+\frac{4}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{\frac{x^2}{x^2}+\frac{5x}{x^2}-\frac{3}{x^2}}}{1+\frac{4}{x}} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{5}{x}-\frac{3}{x^2}}}{1+\frac{4}{x}} = \frac{\sqrt{1}}{1} = 1$. Therefore the horizontal asymptote is $y = 1$.

EXAMPLE: Find the $\lim_{x\to\infty} \arctan x$: This happens when $\lim_{x\to\frac{\pi}{2}} \cos x = 0$. Therefore $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$.

__**NOTE:**__ No function can cross or touch a vertical asymptote, because it is undefined there. A horizontal asymptote can be crossed or touched by the function, in fact, even an infinite number of times. The function need only approach a value $y = L$ at either $\pm\infty$ (or both).

Consider the following example: $f(x) = \frac{1}{x}\sin x$:



EXERCISES:

Find the following limits (if they exist), or show they do not exist:

1) $\lim_{x\to\infty} \frac{1}{x+6}$:

2) $\lim_{x\to\infty} \frac{x+5}{x^2+6x}$:

3) $\lim_{x\to\infty} \frac{3x}{x^2+x}$:

4) $\lim_{x\to-\infty} \frac{x+7}{x-6}$:

5) $\lim_{x\to\infty} \frac{x^2+5x-27}{4x^2+9}$:

6) $\lim_{x\to-\infty} \frac{13x^3-10x+11}{2x^3+6x^2}$:

7) $\lim_{x\to\infty} \frac{x^{\frac{1}{2}}-7}{x^{\frac{1}{4}}+8}$:

8) $\lim_{x\to\infty} \frac{x^9+7x^5+14x^2-12x+4}{x^2+3x+7}$:

9) $\lim_{x\to-\infty} \frac{4x^4}{2x^4+6x^3-7x^2+11}$:

10) $\lim_{x\to\infty} \frac{\sqrt{x^2-x+3}}{x-2}$:

11) $\lim_{x\to\infty} \frac{\sqrt{x^4+3x^2}}{9x^2+7x-5}$:

12) $\lim_{x\to-\infty} \frac{\sqrt{x^2+4}}{2x-2}$:

13) $\lim_{x\to\infty} \frac{\cos x}{x-4}$: (Hint: Use the Squeeze Theorem).

14) $\lim_{x\to\infty} \frac{e^x+1}{e^x-1}$:

15) $\lim_{x\to-\infty} x-3$:

16) $\lim_{x\to\infty} \sqrt{x-100}$:

17) $\lim_{x \to \infty} \ln x$:

Find both the vertical and horizontal asymptotes for the following functions:

18) $f(x) = \frac{4}{2x^2 - 32}$:

19) $f(x) = \frac{x-1}{x^2+5x-6}$:

20) $f(x) = \frac{x^3-5x^2+10}{3x^3-9x^2+6x}$:

21) $f(x) = \frac{x^3+x^2-3x}{13x+9}$:

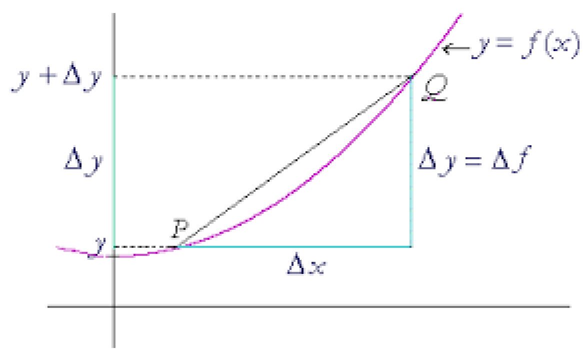
22) $f(x) = \frac{3x^2-1}{5x^2+x-9}$:

CHAPTER 1
SECTION 6

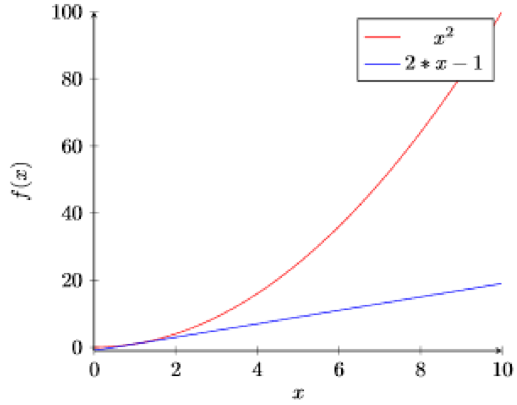
DERIVATIVES

Let's go back to the ideas in Chapter 0 where we discussed rates of change. We started out by talking about velocity. This is a good place to start, because it is a nice and familiar rate of change for most of us. Derivatives are rates of change. We have already calculated average rates of change from our Algebra Days. What we have not done, is calculated instantaneous rates of change. As mentioned before, we need Calculus for this. And we will need limits in order to do this. (Lightbulb Moment!) Why were we doing all those limits, which may have seemed arbitrary and unnecessarily abstract? We will now see this is one such reason. Or that Derivatives are an application of limits, if you would. Most of us do care about things like instantaneous velocity! For example, let us say we are skiing down a mountain. We can already calculate our average velocity down a run on a slope. All we need to know is the distance of the run, and our total time during the run. This is simply $\frac{total\ distance}{total\ time}$. But, what if we wanted to know our exact speed when we were adjacent to, let's say, a particular building. Perhaps we thought we were going especially fast right then. We would need Calculus and limits to determine this.

Let us consider the graph below:



We see the slope of the line that connects P to Q is the average rate of change between P and Q of the curve $y = f(x)$. The line that connects these two points is called the secant line between P and Q. Secant lines connect two points on a graph, and their slope represents an average rate of change. The slope is familiar: $\frac{\Delta y}{\Delta x}$. Let us rename it as $\frac{f(x_2)-f(x_1)}{x_2-x_1}$. What if we wanted to know the instantaneous rate of change at P. Can you see that if we move Q closer and closer to P, that we will get something closer and closer to an instantaneous rate of change? How close can we get? Well, we observe that we can't move Q all the way on top of P, i.e. $P = Q$, or we would be dividing by 0, which we cannot do. So how close can we get without them being equal? Aha! Another lightbulb moment! The limit is what allows us to do that. So we move Q as arbitrarily close as we can to P, with the limit. So the instantaneous rate of change here will be $\lim_{x_2 \to x_1} \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Let us rename this as: $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$: In this form x can be any value, and a is the particular value that we know. The graph below shows us the tangent line at the point (1,1) of the graph $f(x) = x^2$. What is a tangent line? First of all the word "tangent" means touching. A tangent line is a line that touches a graph at a particular point, and shares the same slope as the graph at that point.



DEFINITION: The tangent line to the curve $y = f(x)$ at a point $(a, f(a))$ has the slope

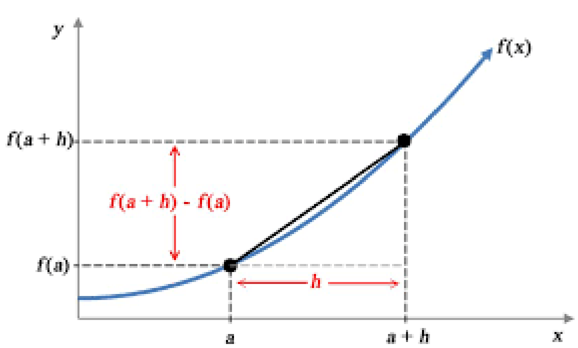
$m = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$, so long as the limit exists.

So the derivative is an instantaneous rate of change. It is the slope of a tangent line at a point. A practical example could also be velocity. Velocity is a rate of change, and the derivative would represent an instantaneous velocity at a point (or a particular time), if we knew a function for the distance.

EXAMPLE: Let $f(x) = x^2$ at the point (2,4). $m = \lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 2}(x+2) = 4.$

Let us redo this problem as a velocity problem. Let $s(t) = t^2$ be the distance function (or change in position) at (2,4) (i.e. at time equals 2 seconds, and distance equals 4 units). Redoing the above problem gives us a velocity of 4, when time is 2 seconds. (This was done as a simple change of variables from x to t.)

We can also rename our familiar points to re-derive the formula into a new form. This should look familiar, as it is the difference quotient that we studied in Algebra class.



Now, we observe the slope here is $m = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$. If we take $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, we again get the slope of the tangent line at $x = a$. Why the rename? It is sometimes easier to manipulate in this form; and we will need this form to find the derivative as a function, instead of at a particular point (which we will do shortly).

EXAMPLE: Let us re-visit our previous example: Let $f(x) = x^2$ at the point (2,4). $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(2+h)^2-f(2)}{h} = \lim_{h\to 0} \frac{4+4h+h^2-4}{h} = \lim_{h\to 0} \frac{h(4+h)}{h} = \lim_{h\to 0} 4 + h = 4$. Just like before. (Note: In the very last step, we took the limit)

$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ is also called the derivative of f at a. Or $f'(a)$. It is the derivative of a function at a particular point.

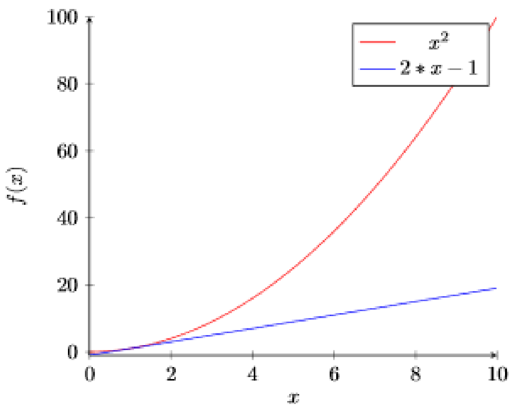
The Derivative as a Function:

Next, we want to derive the derivative (or the slope of the tangent line) for any value of x, rather than only at a particular point. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ becomes $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ where x can be any number, rather than a particular one that we know. This derivative becomes a whole new function, in which we can find the slope of the tangent line (or instantaneous rate of change) at any point. Then, to find the derivative at a particular point; we can substitute any value for x, in the domain of f, that we choose.

EXAMPLE: Let us revisit our example: $f(x) = x^2$. (Only this time, not at any particular point): $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x) = \lim_{h\to 0} \frac{f(x+h)^2-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} 2x + h = 2x$. (Note: During the last step, we actually took the limit, i.e. we substituted 0 in for h. Also note that after we took the limit, we no longer wrote $\lim_{h\to 0} f(x)$. Before that step, it was required, because we hadn't yet taken the limit.)

We can go a bit further with this example, by evaluating it at $x = 2$, and find the same answer as in the two previous examples. $f'(2) = 2 \cdot 2 = 4$, exactly as before. But now, we can substitute any value for x, and get the correct slope at that value.

Let's take this same example and find $f'(1)$. $f'(1) = 2 \cdot 1 = 2$. Now, let's take another look at the graph we had:



Hopefully, you can see at $x = 1$, the slope is 2. We can go even further and find the equation of the tangent line: We use point-slope form. $y - 1 = 2(x - 1) \to y = 2x - 1$, which we can see is the tangent line in our graph.

EXAMPLE: Let $f(x) = x^2 + 5$. Find $f'(x)$. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =$
$\lim_{h\to 0} \frac{(x+h)^2+5-(x^2+5)}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2+5-x^2-5}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = 2x.$

EXAMPLE: Let $f(x) = 2x^2 - 3x + 1$. Find $f'(x)$. $\operatorname{Lim}_{h\to 0} \frac{f(x+h)-f(x)}{h} =$
$\lim_{h\to 0} \frac{2(x+h)^2-3(x+h)+1-(2x^2-3x+1)}{h} = \lim_{h\to 0} \frac{2(x^2+2xh+h^2)-3x-3h+1-2x^2+3x-1}{h} =$
$\lim_{h\to 0} \frac{2x^2+4xh+2h^2-3x-3h+1-2x^2+3x-1}{h} = \lim_{h\to 0} \frac{4xh+2h^2-3h}{h} = \lim_{h\to 0} \frac{h(4x+2h-3)}{h} = 4x - 3.$

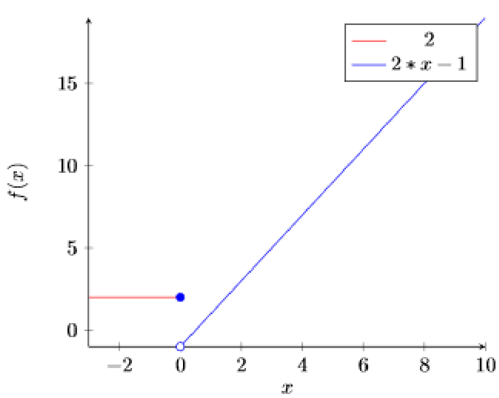
EXAMPLE: Let $f(x) = x^3$. Find $f'(x)$. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3-x^3}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} =$
$\lim_{h\to 0} \frac{h(3x^2+3xh+h^3)}{h} = 3x^2.$

EXAMPLE: Let $f(x) = \frac{1}{x}$. Find $f'(x)$. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} -\frac{\frac{h}{x(x+h)}}{h} =$
$\lim_{h\to 0} -\frac{h}{x(x+h)} \cdot \frac{1}{h} = \lim_{h\to 0} -\frac{1}{x(x+h)} = -\frac{1}{x^2}.$

EXAMPLE: Let $f(x) = \frac{1}{x+2}$. Find $f'(x)$. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+2+h}-\frac{1}{x+2}}{h} = \lim_{h\to 0}\frac{\frac{(x+2)-(x+2+h)}{(x+2+h)(x+2)}}{h} =$
$\lim_{h\to 0} -\frac{\frac{h}{(x+2+h)(x+2)}}{h} = \lim_{h\to 0} -\frac{h}{(x+2+h)(x+2)}\cdot\frac{1}{h} = \lim_{h\to 0} -\frac{1}{(x+2+h)(x+2)} = -\frac{1}{(x+2)^2}$.

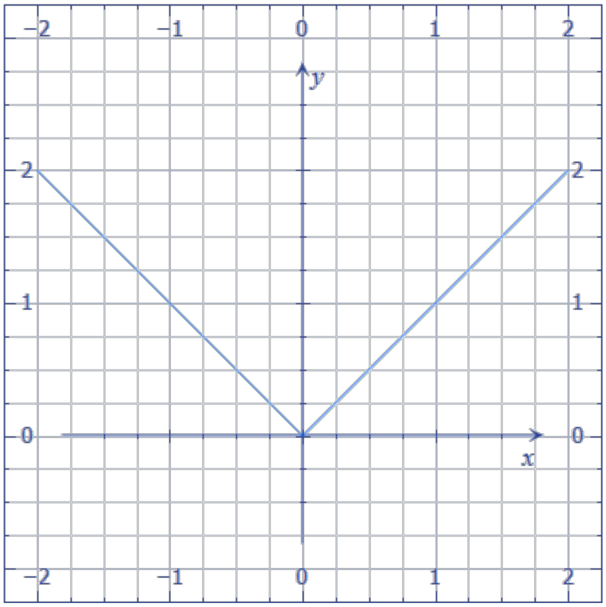
EXAMPLE: Let $f(x) = \sqrt{x-1}$. Find $f'(x)$. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} =$
$\lim_{h\to 0}\frac{(\sqrt{x+h-1}-\sqrt{x-1})}{h} = \lim_{h\to 0}\frac{(\sqrt{x+h-1}-\sqrt{x-1})}{h}\cdot\frac{(\sqrt{x+h-1}+\sqrt{x-1})}{(\sqrt{x+h-1}+\sqrt{x-1})} = \lim_{h\to 0}\frac{x+h-1-(x-1)}{h\,(\sqrt{x+h-1}+\sqrt{x-1})} =$
$\lim_{h\to 0}\frac{h}{h\,(\sqrt{x+h-1}+\sqrt{x-1})} = \frac{1}{2\sqrt{x-1}}$.

DIFFERENTIABILITY: What is differentiability? It means where we are allowed take the derivative. Let's revisit the idea of what the derivative means. It is a rate of change. It is the slope of a tangent line to the graph. Let's look at a previous graph:



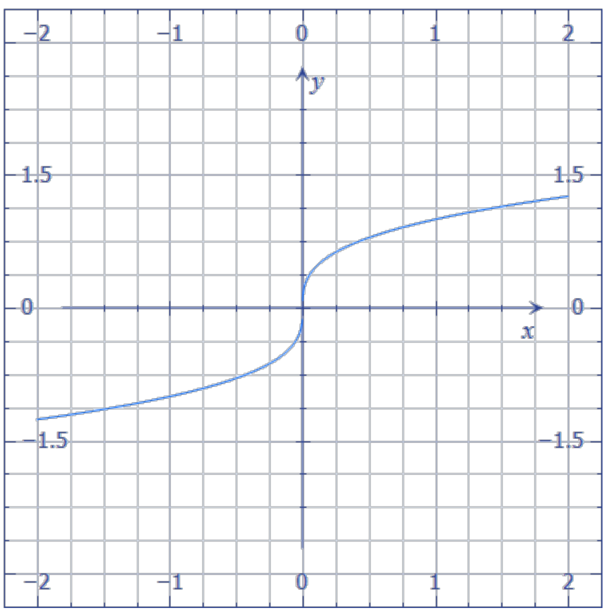
Do you think you could get a tangent line to this graph at $x = 0$? Hopefully, you see that it would be impossible. Recall, that f is not continuous at $x = 0$.

How about this graph?



Do you think you could get a tangent line on this graph at $x = 0$? Again, you can see it wouldn't really work.

What about the graph below:



What about a tangent line at $x = 0$ for this graph? We see that it would be a vertical line. Since the derivative is the slope of the tangent line, we observe that the slope (and therefore, the derivative) would be undefined here.

So for a function to be differentiable, it must be smooth (i.e. no sharp corners), continuous, and with no vertical tangents.

NOTATION: Let's discuss some different notations for the derivative. We already have the derivative as $f'(x)$. We also have $f'(x) = \frac{dy}{dx} = y' = \frac{d(f(x))}{dx}$. Notice the $\frac{dy}{dx} = \frac{d(f(x))}{dx}$ notations were invented by Leibnitz, whom we learned about in Chapter 0. I believe this is the better notation, because it is more explicit about what we are taking the derivative of: It is saying we are taking the derivative of $y = f(x)$ with respect to x. (In this text, we will use these notations interchangeably.)

EXERCISES:

Find $f(a)$ for $f(x)$ for the given value of a, by using i) $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ and ii) $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

1) $f(x) = -x^2,\ x = 3$:
2) $f(x) = x^2 - 3,\ x = 1$:
3) $f(x) = -x^3,\ x = 2$:
4) $f(x) = \frac{1}{x-1},\ x = 4$:
5) Find the equation of the tangent line for the problems in number 1):

Find the derivatives for the following functions: (i.e. find $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.)

6) $f(x) = -x^2$:
7) $f(x) = x^2 - 3$:
8) $f(x) = -x^3$:
9) $f(x) = \frac{1}{x-1}$:
10) $f(x) = 3x^2$:
11) $f(x) = x^2 + 2$:
12) $f(x) = 2x^2 + x + 3$:
13) $f(x) = 5x^2 + 3x - 7$:
14) $f(x) = 3x^2 - 2x - 1$:
15) $f(x) = \frac{2}{x+3}$:
16) $f(x) = \frac{x+1}{x-2}$:
17) $f(x) = \sqrt{x+3}$:
18) $f(x) = \sqrt{2x-2}$:

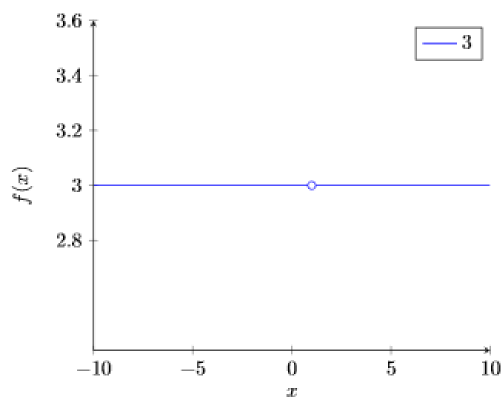
19) $f(x) = \frac{1}{\sqrt{x-1}}$:

20) Let us look at some of the derivatives in problem 2):

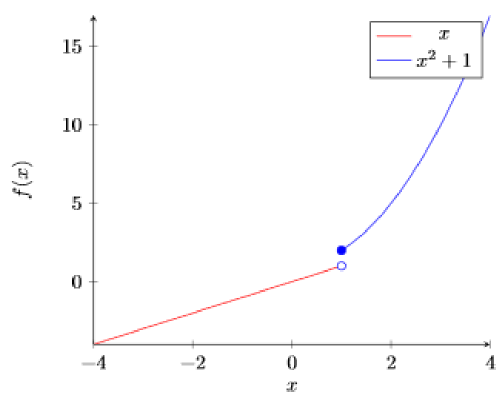
a) Find $f'(-1), f'(0), and\ f'(1)$ for problems 2) a), b), and c), and then sketch the graphs of each, and draw a tangent line at each of the values given.

b) Find the equation of the tangent line at $x = 1$, for 2) d), e), and f).

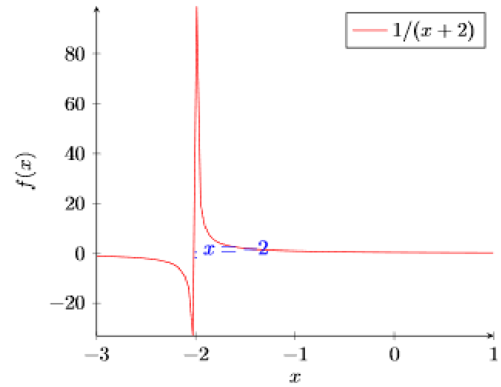
21) State where (meaning at which x-values) the following graphs are not differentiable, and why:



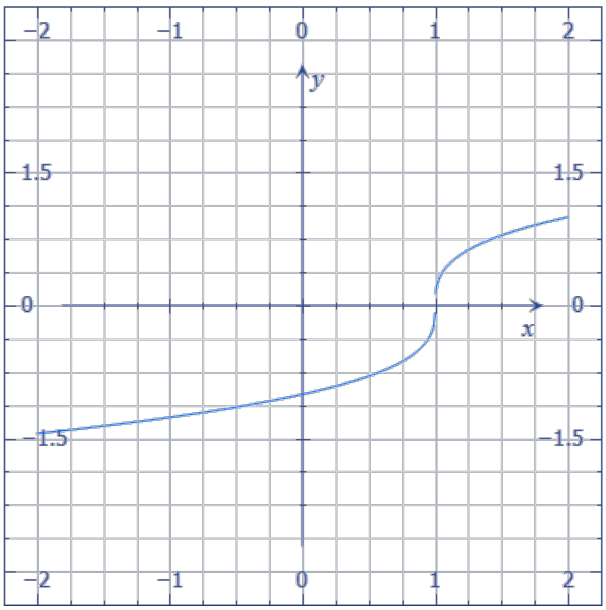
a)



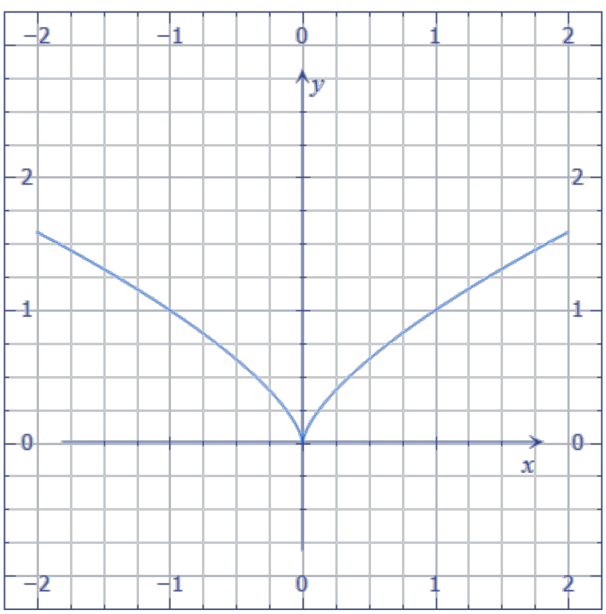
b)



c)



d)



e)

CHAPTER 2
SECTION 1

DIFFERENTIATION
POWER RULE AND SUM/DIFFERENCE RULES:

In the previous chapter, we found the derivative by finding $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. This is the definition. It has worked fine thus far, though we did see it could be a bit heavy on the computational side. Let us look at some of the simplest functions: Power functions, i.e. $f(x) = x^a$. (We will see that even in these simple functions, the definition of the derivative can get quite cumbersome). Let's make a little table:

$f(x)$	$f'(x)$
c	0
x	1
x^2	$2x$
x^3	$3x^2$
$\vdots$	$\vdots$
x^{100}	?
$\vdots$	$\vdots$
x^n	?

Let us compute the first 4 entries of the table, and then we will calculate the derivative for $f(x) = x^{100}$.

We will start with $f(x) = c$. We can easily calculate $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ to obtain the answer. In the interest of preventing tedium, we observe that $f(x) = c$ is a horizontal line. We already know $f'(x)$ is the slope of the tangent line. Since the line is horizontal, the slope is 0 at every point. Therefore $f'(x) = 0$. We can do the same thing with $f(x) = x$. Since it is the line $y = x$, we know the slope is 1 everywhere, so $f'(x) = 1$.

Let us now find $f'(x)$ for $f(x) = x^2$. We already found it was $f'(x) = 2x$. Also, in the previous section, we found for $f(x) = x^3, f'(x) = 3x^2$. So let's move forward to $f(x) = x^{100}$. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^{100}-x^{100}}{h}$. We observe that it was easy enough to write down and begin. What would we have to do next? We understand we would have to expand $(x+h)^{100}$. This is certainly possible, but it would not be fun or efficient.

We hope, instead, that perhaps there's a pattern for these power functions, so that we would not have to use the definition each time. Perhaps you see that for each example, the derivative has this property: The exponent comes down and is multiplied by x, and the new exponent is one less than what it was. We guess that if $f(x) = x^n \to f'(x) = nx^{n-1}$. We will leave the proof for now. But in a subsequent section, we will prove it in an easier manner using derivatives of logarithms.
(**Note:** It even works for $f(x) = c$. We can write $f(x) = c = cx^0 \to c \cdot 0x^{-1} = 0$. (We will need the constant multiple rule, which we will prove shortly to fully see it).

EXAMPLE: Let $f(x) = x^3$. Using the power rule: if $f(x) = x^n \rightarrow f'(x) = nx^{n-1} \rightarrow \; f'(x) = 3x^2$.

EXAMPLE:

1) If $f(x) = x^{\frac{1}{3}}$, then $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$
22) If $y = \sqrt{x} \rightarrow \; y = x^{\frac{1}{2}}$, then $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
23) $\frac{d\left(\sqrt[4]{x^3}\right)}{dx} = \frac{3}{4}x^{-\frac{1}{4}}$
24) If $f(x) = x^{-\frac{5}{2}}$, then $f'(x) = -\frac{5}{2}x^{-\frac{7}{2}}$

CONSTANT MULTIPLE RULE: If c is a constant, and f is a differentiable function then:

If $F(x) = cf(x), then\; F'(x) = cf'(x)$.

PROOF: How will we prove this? We will go back to the definition of the derivative. Where $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. Let $F(x) = cf(x)$. $F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{c} =$
$c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ (by limit law 2, chapter 1) $= cf'(x)$.

EXAMPLE:

1) If $f(x) = 3x^2 \rightarrow f'(x) = \frac{3d(x^2)}{dx} = 3 \cdot 2x = 6x$.
2) If $y = 2x^{\frac{1}{2}} \rightarrow \frac{dy}{dx} = 2 \cdot \frac{1}{2}x^{-\frac{1}{2}} = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$.
3) If $f(x) = 4x^9 \rightarrow f'(x) = 4 \cdot 9x^8 = 36x^8$
4) If $y = \frac{1}{3}x^{\frac{2}{3}} \rightarrow \frac{dy}{dx} = \frac{2}{9}x^{-\frac{1}{3}}$.

SUM/DIFFERENCE RULE: If f and g are both differentiable functions, then if

$F(x) = f(x) \pm g(x), then\; F'(x) = f'(x) \pm g'(x)$.

PROOF: Let us prove: If $F(x) = f(x) + g(x)$, then $F'(x) = f'(x) + g'(x)$. (The Sum Rule.) (The Difference rule is left as an exercise for the student to prove on his/her own).

Let $F(x) = f(x) + g(x)$, $F'(x) =$
$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-\left(f(x)+g(x)\right)}{h} = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} =$
$\lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$ (by limit law 1, Chapter 1)
$= f'(x) + g'(x)$.

EXAMPLE: Let $f(x) = x^3 + x^2 + x + 2 \to f'(x) = \frac{d(x^3)}{dx} + \frac{d(x^2)}{dx} + \frac{d(x)}{dx} + \frac{d(2)}{dx} = 3x^3 + 2x + 1 + 0 = 3x^3 + 2x + 1.$

EXAMPLE: Let's combine it with the constant multiple rule: Let $f(x) = 4x^2 + \frac{1}{2}x + 7 \to f'(x) = \frac{4d(x^2)}{dx} + \frac{\frac{1}{2}d(x)}{dx} + \frac{d(1)}{dx} = 4 \cdot 2\text{x} + \frac{1}{2} \cdot 1 + 0 = 8x + \frac{1}{2}.$

EXAMPLE:

1) Let $f(x) = 2x^3 + 4x^2 + \frac{1}{3}x - 100 \to \ f'(x) = 2 \cdot 3x^2 + 4 \cdot 2x + \frac{1}{3} \cdot 1 - 0 = 6x^2 + 8x + \frac{1}{3}.$
2) Let $f(x) = 2x^{\frac{2}{3}} - 5x^{10} - \sqrt{x} \to \ f'(x) = \frac{4}{3}x^{-\frac{1}{3}} - 50x - \frac{1}{2}x^{-\frac{1}{2}}.$
3) Let $y = 9\sqrt[3]{x^5} + \frac{4}{x^3} + 2x^6 = 9x^{\frac{5}{3}} + 4x^{-3} + 2x^6 \to \ \frac{dy}{dx} = 15x^{\frac{2}{3}} - 12x^{-4} + 12x^5$
4) Let $f(x) = \frac{3x^2+2x-1}{x} = 3x + 2 - x^{-1} \to \ f'(x) = 3 - x^{-2} = 3 - \frac{1}{x^2}$

EXAMPLE: Find the equation of the tangent line to the graph $y = 3x^2 - 9$ at the point (1,6):
Let us first find the slope at $x = 1$: We must first find the derivative: $\frac{dy}{dx} = 6x$. At $x = 1$: $\frac{dy}{dx} = 6$. We now use point-slope form: $y - y_1 = m(x - x_1) \to \ y - 6 = 6(x - 1) \to y = 6x.$

EXAMPLE: Find the equation of the tangent line to the graph $f(x) = x^3 - 2x$ at $x = 2$.
$f'(x) = 3x^2 - 2 \to \ f'(2) = 10$. In this case we also need to find $f(2) = 4$. So the equation of the tangent line is: $y - 4 = 10(x - 2) \to \ y = 10x - 16.$

HIGHER-ORDER DERIVATIVES: What is a higher-order derivative? A higher order derivative is a derivative of a derivative. For example, a second derivative is simply the derivative of the first derivative. The third derivative is the derivative of the second derivative, etc.

Let us introduce some notation:

Function	$f(x)$	y	
First Derivative	$f'(x)$	$\frac{dy}{dx}$	y'
Second Derivative	$f''(x)$	$\frac{d^2y}{dx^2}$	y''
Third Derivative	$f'''(x)$	$\frac{d^3y}{dx^3}$	y'''
Fourth Derivative	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$y^{(4)}$
⋮	⋮	⋮	⋮
Nth Derivative	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$y^{(n)}$

EXAMPLE: Let $f(x) = 3x^4 + 2x^2 + 9x + 12$. Find $f^{(5)}(x)$.

$f'(x) = 12x^3 + 4x^2 + 9$
$f''(x) = 36x^2 + 8x$
$f'''(x) = 72x + 8$
$f^{(4)}(x) = 72$
$f^{(5)}(x) = 0$

VELOCITY AND ACCELERATION: Recall that we discussed velocity and acceleration in Chapter 0, and again in Chapter 1, Section 6.
We discovered that velocity was the derivative of distance (or change in position). Acceleration is the derivative of velocity, or second derivative of distance.

EXAMPLE: Let $s(t) = 5t^2 - 10t + 12$. Find the velocity and acceleration at $t = 2$ seconds. Let $s(t)$ represent distance in meters.

$s'(t) = v(t) = 10t - 10$. At $t = 2, v(2) = 20 - 10 = 10\frac{m}{s}$.
$a(t) = v'(t) = s''(t) = 10$. Therefore, $a(2) = 10\frac{m}{s^2}$. (Note that acceleration here is constant. Gravity is also an example of constant acceleration: $9.8\frac{m}{s^2}$.)

EXERCISES:

Find the following derivatives by using The Power Rule, The Constant Multiple Rule, The Sum/Difference Rule, or any combination thereof:

1) $f(x) = x^5$:

2) $f(x) = x^{\frac{1}{5}}$:

3) $y = \frac{1}{x^3}$

4) $y = \sqrt{x^7}$

5) $f(x) = x^{-\frac{5}{6}}$

6) $f(x) = 3x^{\frac{3}{2}}$

7) $f(x) = \frac{1}{2}x^{-\frac{1}{2}}$

8) $f(x) = 5\sqrt[7]{x^2}$

9) $f(x) = x^6 + 4x^5 - 7x^3 + 6x - 12$

10) $f(x) = 2x^3 - 6x^2 + 3x - 7$

11) $f(x) = 4x^{\frac{1}{2}} - 12x^5 - \frac{1}{x^4} + \frac{3}{x^2} - 2x^3$

12) $f(x) = 20x^2 - \frac{7}{x^{\frac{1}{3}}} + \frac{3}{\sqrt[3]{x^4}}$

13) $f(x) = (x^2 - 1)(x + 4)$

14) $f(x) = (x + 1)(2x - 9)$

15) $f(x) = x^4(x^2 + 2x - 7)$

16) $f(x) = (x^2 - x - 1)(x + 6)$

17) $f(x) = \frac{x^{\frac{1}{2}} - 2x^2 + 4x + 6}{x^2}$

18) $f(x) = \frac{(x+1)(x-1)}{\sqrt{x}}$

Find the equation of the tangent line to the curve at the given point (or value).

19) $f(x) = x^2 + 9,\ (1,10)$

20) $f(x) = x^3 + 1,\ (2,9)$

21) $f(x) = \sqrt{x},\ x = 4$

22) $f(x) = \frac{1}{\sqrt{x^3}},\ x = 4$

Find the following higher order derivatives:

23) Find $f^{(5)}(x)$ for $f(x) = x^6 - 2x^5 + 3x^4 + 6x - 9$

24) Find $\frac{d^4y}{dx^4}$ for $y = 3x^5 - 20x + \frac{1}{x} + 6$

25) Find $\frac{d^2y}{dx^2}$ for $y = x^{\frac{1}{2}} - \frac{3}{x} + x$

26) Find $f''(x)$ for $f(x) = \frac{1}{x^2} + 12x^2 + \sqrt{x}$

27) Let $s(t) = 2t^2 + 4t - 2$ by the distance function where s is in meters, and t is in seconds.

a) Find the velocity and acceleration when $t = 1$ second

b) Find the velocity and acceleration when $t = 3$ seconds

CHAPTER 2
SECTION 2

DIFFERENTIATION
DERIVATIVES OF THE NATURAL EXPONENTIAL FUNCTION AND THE TRIGONOMETRIC FUNCTIONS OF SINE AND COSINE

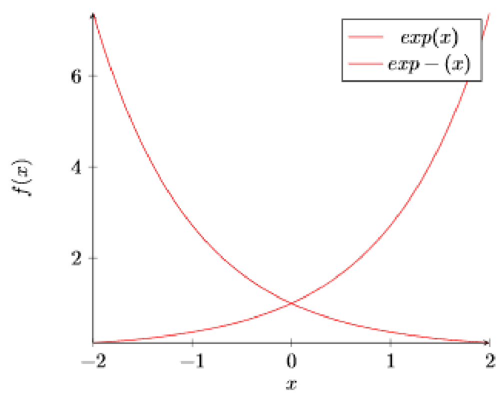
EXPONENTIAL FUNCTIONS AND THEIR DERIVATIVES:

Let us recall what an exponential function is: $f(x) = a^x, a \neq 1, a > 0$.

(**Note**: Many of my students confuse $f(x) = a^x$ with $f(x) = x^a$ (a power function, not exponential). Students often see a base and an exponent, and think they are the same thing. I have found it to be common for Calculus students to try and perform the derivative of $f(x) = a^x$ using the power rule (as we used in Section 1 for power functions of the form $f(x) = x^a$, which does not apply here). We note that the exponent is varying in an exponential function, with a fixed base. And, the base is varying with a fixed exponent in the power function).

$f(x) = a^x, a \neq 1, a > 0 \rightarrow \ x \neq 0$, or 1, (because they would be constant functions, rather than exponential functions). And let us also recall that $x < 0$ would give us complex numbers, which we do not consider in a Calculus course.

Recall the graphs:



Recall when $a > 1, f(x)$ is increasing. And when $0 < a < 1, f$ is decreasing.

Let us now **derive the derivative**: We return to the definition: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} =$
$\lim_{h \to 0} \frac{a^{x+h}-a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$ (Note: we can do that last step because there is no h in a^x, so it does not involve a limit as $h \rightarrow 0$.

We observe the derivative of $f(x) = a^x$ is: $a^x \lim_{h\to 0} \frac{a^h - 1}{h}$, which we see is itself times $\lim_{h\to 0} \frac{a^h - 1}{h}$. So what about this limit: $\lim_{h\to 0} \frac{a^h - 1}{h}$? We want to see if perhaps there is a number a, such that $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$, so that the derivative of $f(x) = a^x$ is itself.

Let's try the number 2! (Why? Because these ideas have already been derived, and we already have an idea of where the number will lie). We see that if we substitute in 2 for a, we have a form of $\frac{0}{0}$. So how will we attempt this? We will not find the limit explicitly, but instead, we will approximate it with a calculator. We substitute in smaller and smaller values for h, and we see that when $a \approx 2$, $\lim_{h\to 0} \frac{a^h - 1}{h} \approx .7$. Okay, so that is a little too small. Let's try $a = 3$. Again we have a form of $\frac{0}{0}$, so we approximate with a number very close to zero for h. When $a \approx 3$, $\lim_{h\to 0} \frac{a^h - 1}{h} \approx 1.1$. Aha! So we know it would have to be a number between 2 and 3. In fact, the number is precisely $e = 2.718 \ldots$ We also recognize this as the base for the natural exponential function $f(x) = e^x$. Therefore, when $f(x) = e^x$, $f'(x) = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$.

So when $f(x) = \boldsymbol{e^x} \rightarrow \boldsymbol{f'(x) = e^x}$. f is its own derivative. When you look at the above graph, you see the slope of the tangent line everywhere on f is exactly f.

THREE DEFINITIONS OF THE NUMBER $\boldsymbol{e}$:

1) e is such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.

2) $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.718 \ldots$

3) $e = \lim_{h\to 0} (1 + h)^{\frac{1}{h}} = 2.718 \ldots$

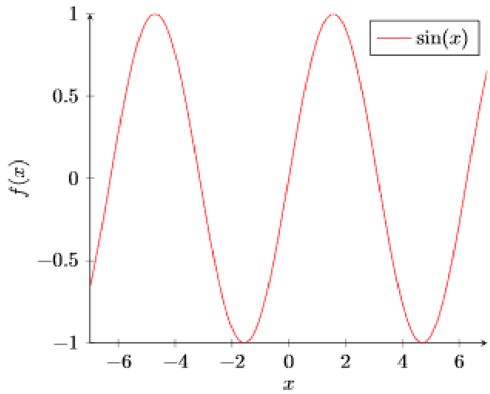
EXAMPLE:

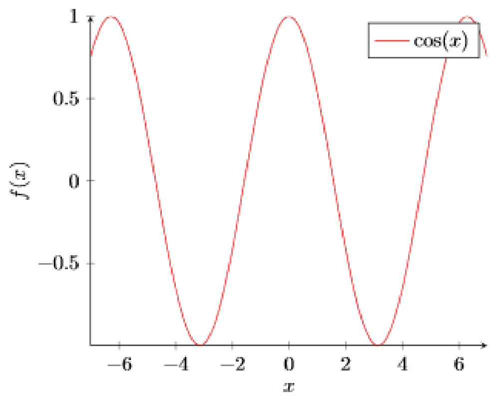
1) $f(x) = 2e^x \rightarrow f'(x) = 2e^x$
2) $f(x) = \frac{1}{2}e^x + 3x^2 \rightarrow f'(x) = \frac{1}{2}e^x + 6x$

EXAMPLE: Let us find the equation of the tangent line for $f(x) = 3e^x$ at the point (0,3):
$f'(x) = 3e^x \rightarrow\ f'(0) = 3e^0 = 3$. Then $y - 3 = 3(x - 0) \rightarrow y = 3x + 3$.

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS:

Let us start with the familiar graphs of $f(x) = \sin x$ and $f(x) = \cos x$:





Let us recall what the derivative means. It is the slope of the tangent line at every point on the graph. Look at the above graphs, and observe that wherever the slope of $f(x) = \sin x$ is 0, that the cosine graph goes through the x-axis. This gives us the idea that $f(x) = \cos x$ might be the derivative of $f(x) = \sin x$.

Let us prove this using the definition of the derivative again:

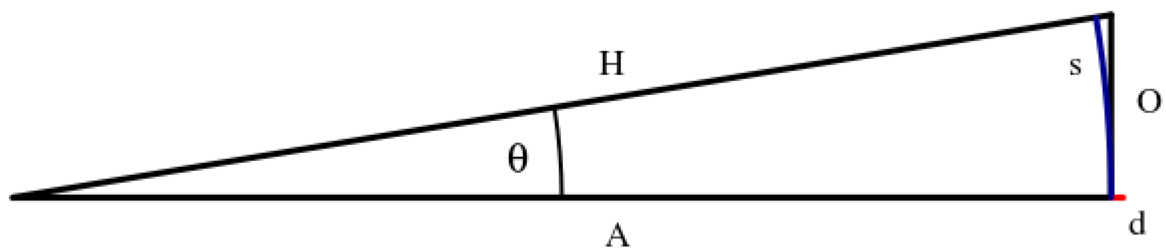
$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h-\sin x}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h}$ (by limit law 1) = $\sin x\lim_{h\to 0}\frac{\cos h-1}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h}$ (by the fact that $\sin x$, $\cos x$ do not have an h, therefore they are not involved in the limit).

Let's summarize: If $f(x) = \sin x \to f'(x) = \sin x\lim_{h\to 0}\frac{\cos h-1}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h}$.

To finish this proof, we have to calculate the two limits: $\lim_{h\to 0}\frac{\cos h-1}{h}$, and $\lim_{h\to 0}\frac{\sin h}{h}$.

Let's look at $\lim_{h\to 0}\frac{\sin h}{h}$ first. I will not prove this now. Instead, I will show you the basic idea (and leave the proof for your own exploration. It is a rather detailed geometric proof).

Let us observe that $\lim_{h \to 0} \frac{\sin h}{h} \approx 1$ by inspection.



Take a look at the above triangle. In physics we use what we call the small angle approximation for $\sin\theta$. I.e., for very small angles θ, $\sin\theta \approx \theta$. Assume the above triangle has a very small angle θ.

$$\sin\theta = \frac{O}{H}, \tan\theta = \frac{O}{A}, \qquad O \approx s, \qquad H \approx A \to \sin\theta = \frac{O}{H} \approx \frac{O}{A} = \tan\theta \approx \frac{s}{A} = \frac{A\theta}{A} = \theta.$$

This implies $\sin\theta = \theta$. (We call this "proof by picture": Meaning it is not a formal proof, but does convey the meaning in a more visual, though not formal way). If $\sin\theta = \theta$, for θ close to 0, then $\lim_{h \to 0} \frac{\sin h}{h} \approx 1$. The formal proof would indeed give us $\lim_{h \to 0} \frac{\sin h}{h} = 1$.

Now we have the $\lim_{h \to 0} \frac{cosh-1}{h}$. We will formally prove this one: $\lim_{h \to 0} \frac{cosh-1}{h} = \lim_{h \to 0} \frac{cosh-1}{h} \cdot \frac{cosh+1}{cosh+1} = \lim_{h \to 0} \frac{\cos^2 h-1}{h(cosh+1)} = \lim_{h \to 0} \frac{-\sin^2 h}{h(cosh+1)} = -\lim_{h \to 0} \frac{sinh}{(cosh+1)} \cdot \frac{sinh}{h} = -\lim_{h \to 0} \frac{sinh}{(cosh+1)} \cdot \lim_{h \to 0} \frac{sinh}{h} = (-1)\frac{0}{2} \cdot 1 = 0.$
Therefore, $\lim_{h \to 0} \frac{cosh-1}{h} = 0$.

We now conclude that for $f(x) = \sin x, f'(x) = \sin x \lim_{h \to 0} \frac{\cos h-1}{h} + cosx \lim_{h \to 0} \frac{\sin h}{h} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x$.

Therefore the derivative of $f(x) = \sin x = \cos x$.

If $f(x) = \cos x, f'(x) = \lim_{h \to 0} \frac{\cos(x+h)-\cos x}{h} = -\sin x$ (by a similar proof as the one above).

(Note: For the derivatives of the other trigonometric functions, we will need the quotient rule, which we will discover in the next section).

<u>SUMMARY:</u>

1) $f(x) = e^x \to f'(x) = e^x$
2) $f(x) = \sin x \to f'(x) = \cos x$
3) $f(x) = \cos x \to f'(x) = -\sin x$

EXAMPLE:

1) Let $f(x) = 2\cos x - \frac{1}{2}\sin x + x^2 + 3e^x$. $f'(x) = -2\sin x - \frac{1}{2}\cos x + 2x + 3e^x$:

2) Let $f(x) = \sqrt{2}\sin x - 4\cos x + 4x^3$. $f'(x) = \sqrt{2}\cos x + 4\sin x + 12x^2$

3) Let $f(x) = 2\sin x - \cos x$. Find the equation of the tangent line when $x = \pi$. First find $f'(x) = 2\cos x + \sin x$. When $x = \pi, f'(\pi) = -2 + 0 = -2$. Next, $f(\pi) = 0 + 1 = 1$. Then we have $y - 1 = -2(x - \pi) = y = -2x + 2\pi + 1$.

EXAMPLE:

1) Find $\lim_{x \to 0} \frac{2\sin x}{3x} = \frac{2}{3} \lim_{x \to 0} \frac{\sin x}{x} = \frac{2}{3} \cdot 1 = \frac{2}{3}$

2) Find $\lim_{x \to 0} \frac{\sin 2x}{4x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin 2x}{2x}$. Let $u = 2x \to$ $\frac{1}{2} \lim_{x \to 0} \frac{\sin 2x}{2x} = \frac{1}{2} \lim_{\frac{u}{2} \to 0} \frac{\sin u}{u} = \frac{1}{2} \cdot 1 = \frac{1}{2}$

EXERCISES:

Find the following derivatives:

1) $f(x) = 12e^x - 4\sin x + \frac{1}{x^{\frac{4}{3}}}$:

2) $f(x) = \frac{1}{\sqrt{2}}\cos x - 5e^x + 12\sin x - \sqrt[4]{x}$:

3) $f(x) = 3\sin x + 14e^x$

Find the equation of the tangent line to the curve at the given x-value:

4) $f(x) = 2e^x - \sin x\,, x = 0$:

5) $f(x) = \cos x - \sin x\,, x = 2\pi$

6) $f(x) = \frac{1}{2}e^x, x = 2$:

Find the following limits:

7) $\lim_{x \to 0} \frac{3\sin x}{8x}$

8) $\lim_{x \to 0} \frac{\sin 3x}{8x}$

CHAPTER 2
SECTION 3

DIFFERENTIATION
PRODUCT AND QUOTIENT RULES
OTHER TRIGONOMETRIC DERIVATIVES

PRODUCT RULE:

The product rule is used for, well products! (I.e. multiplication). Let's start with a function that can be written as a very simple product: $f(x) = x^3 \cdot x^2$. We already know how to do this derivative. We simply multiply, and then take the derivative. $f(x) = x^3 \cdot x^2 = x^5 \rightarrow f'(x) = 5x^4$. Intuitively, because of what we already know from the sum/difference rule and the constant multiple rule, we are tempted to think the derivative of fg is $f'g'$. Let's try that. If $f(x) = x^3 \cdot x^2$, $f'(x)$ would be $3x^2 \cdot 2x = 6x^3 \neq f'(x)$ by using this false method. So we instantly see it does not work. (To disprove any theorem, we need only provide one counterexample). In fact $\frac{d(fg)}{dx} = fg' + f'g$.

Let's try our example again using the product rule: $f(x) = x^3 \cdot x^2 \rightarrow f'(x) = x^3 \cdot 2x + 3x^2 \cdot x^2 = 5x^4$. So we see it works in this one example.

PROOF: Let $F(x) = f(x)g(x) \rightarrow F'(x) = \lim\limits_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim\limits_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} =$ $(\text{Warning: Trick!}) = \lim\limits_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)+\boldsymbol{f(x+h)g(x)-f(x+h)g(x)}}{h} =$ $\lim\limits_{h\to 0} \frac{f(x+h)[g(x+h)-g(x)]}{h} + \lim\limits_{h\to 0} \frac{g(x)[f(x+h)-f(x)]}{h} = \lim\limits_{h\to 0} f(x+h) \cdot \lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h} +$ $g(x) \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = f(x)g'(x) + g(x)f'(x)$. (**Note:** The trick you might not intuit is in red),

Summary: If $F(x) = f(x)g(x) \rightarrow F'(x) = f(x)g'(x) + f'(x)g(x)$.

Okay, you might not want to label the function's products with f and g, and then have to remember which one is which. A simple way to remember it is:

The first times the derivative of the second, plus the derivative of the first times the second.

It is much easier to see the first and the second upon a quick inspection.

EXAMPLES:

1) If $f(x) = (x^2 + 9x)(3x^3 + 12x^2)$, then $f'(x) = (x^2 + 9x)(9x^2 + 24x) + (2x + 9)(3x^3 + 12x^2)$. (Note: $(x^2 + 9x)$ is the first, and $(3x^3 + 12x^2)$ is the second).

2) If $f(x) = (3x^2 + 4x - 10)\left(x^{\frac{1}{2}} - 2x^{-2} - 4x^3\right) \rightarrow f'(x) =$ $(3x^2 + 4x - 10)\left(\frac{1}{2\sqrt{x}} + \frac{4}{x^3} - 12x^2\right) + (6x + 4)\left(x^{\frac{1}{2}} - 2x^{-2} - 4x^3\right)$. (Note: $(3x^2 + 4x - 10)$ is first, $\left(x^{\frac{1}{2}} - 2x^{-2} - 4x^3\right)$ is second.

3) Let $f(x) = e^x \sin x \rightarrow \quad f'(x) = e^x \cos x + e^x \sin x$

4) Let $f(x) = \sin x \cos x \rightarrow \quad f'(x) = \sin x\,(-\sin x) + \cos x \cos x = \cos^2 x - \sin^2 x.$

5) Let $f(x) = e^x(3x^3 - \sin x) \rightarrow f'(x) = e^x(9x^2 - \cos x) + e^x(3x^3 - \sin x).$

6) Let $f(x) = (4x^3 + 3x^2 - 10x)\cos x \rightarrow f'(x) = (4x^3 + 3x^2 - 10x)(-\sin x) + (12x^3 + 6x - 10)\cos x = -(4x^3 + 3x^2 - 10x)(\sin x) + (12x^3 + 6x - 10)\cos x.$

7) Find $\frac{d^2y}{dx^2}$ for the following functions:

a) $y = e^x \cos x \rightarrow \frac{dy}{dx} = -e^x \sin x + e^x \cos x \rightarrow \frac{d^2y}{dx^2} = -e^x \cos x - e^x \sin x - e^x \sin x + e^x \cos x = -2e^x \sin x.$

b) $y = (x^2 + 1)^2 = (x^2 + 1)(x^2 + 1) \rightarrow \frac{dy}{dx} = (x^2 + 1)(2x) + 2x(x^2 + 1) = 4x(x^2 + 1) \rightarrow \frac{d^2y}{dx^2} = 4x(2x) + 4(x^2 + 1) = 8x^2 + 4x^2 + 4 = 12x^2 + 4.$

8) Find the equation of the tangent line for the following functions at the given x-value:

a) $y = (3x^2 + 9)(2x^3 - 2x)$ at $x = 1$. First we find $\frac{dy}{dx} = (3x^2 + 9)(6x^2 - 2) + 6x(2x^3 - 2x)$. At $x = 1$: $\frac{dy}{dx} = 12 \cdot 4 + 6 \cdot 0 = 48$. At $x = 1, y = 0$. Therefore, the equation of the tangent line is $y = 48(x - 1) \rightarrow y = 48x - 48$.

b) $f(x) = e^x \sin x + e^x \cos x\,, at\ x = 0.\ f'(x) = e^x \cos x + e^x \sin x - e^x \sin x + e^x \cos x = 2e^x \cos x.\ f'(0) = 2,\ f(0) = 1 \rightarrow\ y - 1 = 2(x - 0) \rightarrow y = 2x + 1.$

QUOTIENT RULE:

This rule is used for quotients (division).

If we didn't just introduce the product rule, you might intuitively assume that if $F(x) = \frac{f(x)}{g(x)}$, $F'(x) = \frac{f'(x)}{g'(x)}$. This would again be incorrect. Let's use a simple example. Let $f(x) = \frac{x^5}{x^2}$. We can do this by simplifying and using the Power Rule. $f(x) = \frac{x^5}{x^2} = x^3 \rightarrow f'(x) = 3x^2$. If we tried the method we immediately thought of we would get $f'(x) = \frac{5x^4}{2x} = \frac{5}{2}x^3$, which we observe is incorrect. Therefore, we can conclude that if $F(x) = \frac{f(x)}{g(x)},\ F'(x) \neq \frac{f'(x)}{g'(x)}$.

In fact, the quotient rule is as follows: If $F(x) = \frac{f}{g} \to F'(x) = \frac{gf' - g'f}{g^2}$.

Again, you might not want to label your function using f and g, and have to remember which one is which. So in laymen's terms:

$$\frac{\boldsymbol{Bottom\ times\ derivative\ of\ Top - derivative\ of\ Bottom\ times\ Top}}{(\boldsymbol{Bottom})^2}$$

Or, as my students have taught me: $\frac{\boldsymbol{Lo\ d\ Hi - Hi\ d\ Lo}}{\boldsymbol{Lo\ Lo}}$.

Let's go back to our original example: $f(x) = \frac{x^5}{x^2} = x^3 \to f'(x) = 3x^2$. Let us now apply the Quotient Rule to see that it works in this particular case. Let x^5 be the top, and x^2 be the bottom. Then, $f'(x) = \frac{x^2 \cdot 5x^4 - 2x \cdot x^5}{(x^2)^2} = \frac{5x^6 - 2x^6}{x^4} = \frac{3x^6}{x^4} = 3x^2$. And, we get the correct answer.

The proof of this one is very similar to the proof of the Product Rule, and is left as an exercise for the student.

EXAMPLE:

1) Let $f(x) = \frac{2x+4}{x^2-1}$. We will call $x^2 - 1$ the bottom, and $2x + 4$ the top. Then $f'(x) = \frac{(x^2-1)2 - 2x(2x+4)}{(x^2-1)^2} = \frac{2x^2 - 2 - 4x^2 - 4x}{(x^2-1)^2} = \frac{-2x^2 - 4x - 2}{(x^2-1)^2}$.

2) Let $f(x) = \frac{x^2+2x-9}{4x^2+7}$. Let $x^2 + 2x - 9$ be the top, and $4x^2 + 7$ be the bottom. Then $f'(x) = \frac{(4x^2+7)(2x+2) - 8x(x^2+2x-9)}{(4x^2+7)^2} = \frac{4x^3 + 8x^2 + 14x + 14 - 8x^3 - 16x^2 - 72x}{(4x^2+7)^2} = \frac{-4x^3 - 8x^2 - 64x + 14}{(4x^2+7)^2}$.

3) Let $y = \frac{x^{-3} + \sqrt{x}}{e^x + 2x} \to \frac{dy}{dx} = \frac{(e^x+2x)\left(-3x^{-4} + \frac{1}{2}x^{-\frac{1}{2}}\right) - (e^x+2)(x^{-3} + x^{\frac{1}{2}})}{(e^x+2x)^2}$ (We will leave the simplifying to the reader on this one).

EXAMPLE:

Find $\frac{d^2y}{dx^2}$ for $y = \frac{e^x+1}{x-2}$. $\frac{dy}{dx} = \frac{(x-2)e^x - (e^x+1)}{(x-2)^2} = \frac{xe^x - 2e^x - e^x - 1}{(x-2)^2} = \frac{xe^x - 3e^x - 1}{x^2 - 4x + 4} \to \frac{d^2y}{dx^2} = \frac{(x^2-4x+4)(xe^x + e^x - 3e^x) - (2x-4)(xe^x - 3e^x - 1)}{(x^2-4x+4)^2}$. (**Note:** The red indicates use of the product rule).

EXAMPLE:

Find the equation to the tangent line at the following point: $y = \frac{x^2+1}{x-2}$ at $(1, -2)$.

$\frac{dy}{dx} = \frac{(x-2)2x-(x^2+1)}{(x-2)^2} = \frac{2x^2-4x-x^2-1}{(x-2)^2} = \frac{x^2-4x-1}{(x-2)^2}$. At $x = 1$: $\frac{dy}{dx} = -\frac{4}{1} = -4$. So the equation of the tangent line to curve at $(1, -2)$ is: $y + 2 = -4(x - 1) \rightarrow y = -4x + 2$.

OTHER TRIGONOMETRIC DERIVATIVES:

Now that we have the Quotient Rule, we can easily prove the derivatives of the other Trigonometric Functions without having to use the definition of the derivative.

1) $\frac{d(\tan x)}{dx} = \sec^2 x$

2) $\frac{d(\sec x)}{dx} = \sec x \tan x$

3) $\frac{d(\csc x)}{dx} = -\csc x \cot x$

4) $\frac{d(\cot x)}{dx} = -\csc^2 x$

PROOF of number 1)

$f(x) = \tan x = \frac{\sin x}{\cos x}$. Using the Quotient Rule, we get $f'(x) = \frac{\cos x \cos x-(-\sin x) \sin x}{\cos^2 x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$.

The other 3 can easily be proven using the Quotient Rule and are left as an exercise for the student.

SUMMARY:

1) $\boldsymbol{\frac{d(\sin x)}{dx} = \cos x}$

2) $\boldsymbol{\frac{d(\cos x)}{dx} = -\sin x}$

3) $\boldsymbol{\frac{d(\tan x)}{dx} = \sec^2 x}$

4) $\boldsymbol{\frac{d(\sec x)}{dx} = \sec x \tan x}$

5) $\boldsymbol{\frac{d(\csc x)}{dx} = -\csc x \cot x}$

6) $\boldsymbol{\frac{d(\cot x)}{dx} = -\csc^2 x}$

EXAMPLE:

1) If $f(x) = \frac{\cos x - 1}{\sin x + \tan x} \rightarrow \ f'(x) = \frac{(sinx+tanx)(-sinx)-(cosx+\sec^2 x)(cosx-1)}{(sinx+tanx)^2}$

2) Let $y = \frac{e^x \sin x}{1 - \sec x} \rightarrow \frac{dy}{dx} = \frac{(1-secx)(e^x cosx + e^x sinx)-(-secxtanx)(e^x sinx)}{(1-secx)^2}$

3) Let $f(x) = e^x \tan x + x^2 \cot x \rightarrow \ f'(x) = e^x \sec^2 x + e^x \tan x + x^2(-\csc^2 x) + 2x \cot x$

EXERCISES:

Differentiate:

1) $f(x) = x^2(2x^3 + 4x^2 + 5x - 6)$
2) $f(x) = (4x^2 + 9x - 1)(3x^4 + 6x^2 - 10x + 7)$
3) $f(x) = (e^x + 10x^2 - 4x)(\sin x - 5x^2)$
4) $y = (20x^2 + \tan x - \frac{1}{\sqrt{x}})(\sec x - \cos x)$
5) $y = (\csc x + \frac{1}{2}e^x + \frac{1}{x^{\frac{1}{7}}})(2\sin x + \cot x + 3x^4)$
6) $y = (3x^3 + 4x^2 - 2x + 8)(4e^x + x^x - \sin x)$
7) $f(x) = \frac{x^2+2}{2x-4}$
8) $f(x) = \frac{4x^3-12x+7}{x^2+4x+9}$
9) $y = \frac{e^x-2x^2+x^{-3}}{4x^2+9x}$
10) $y = \frac{2\sqrt{x}-3x^3+\frac{1}{x}}{e^x+\sin x-x}$
11) $f(x) = \frac{\tan x+\cos x}{e^x-\csc x}$
12) $f(x) = \frac{5x^2+4e^x-\sin x}{x^{-2}+\cot x}$
13) $y = \frac{e^x-\sin x}{e^x\cos x+4x^4}$
14) $f(t) = \frac{t^2+2t-\sec t}{\sqrt{t}+4\sin t}$
15) $y = \frac{1}{t^2+\sin t+t^{\frac{2}{3}}}$

16) $y = \frac{x}{e^x - x\sin x}$

17) $f(x) = \frac{cx}{c\sin x - cx^4 + cx}$ where c is a constant.

18) $y = \frac{9x^2 + \frac{1}{4}e^x}{\sqrt{2}x^2 - 4x^5 + \frac{1}{x^3}}$

19) $f(x) = \frac{12x^3}{e^x + 20x}$

Find $\frac{d^2y}{dx^2}$ for the following functions:

20) $f(x) = \sin x \cos x$

21) $y = \frac{x^2+4}{4x-9}$

22) $y = e^x \cos x + x\sin x$

23) $f(x) = \frac{x^2}{2e^x + 3x}$

Find the equation of the tangent line at the following points:

24) $f(x) = (x^2 + 2x - 3)(2x + 6), (1,0)$

25) $y = e^x \sin x - 3\cos x\,, (0, -3)$

26) $f(x) = \frac{1-2x}{x^2+2},\ (2, -\frac{1}{2})$

Challenge Problems:

Prove the quotient rule, and the derivatives for the rest of the trigonometric functions.

CHAPTER 2
SECTION 4

DIFFERENTIATION
CHAIN RULE

What is the Chain Rule used for? It is used for composite functions, i.e. $f \circ g = f\big(g(x)\big)$. For example, let $F(x) = (x^2 + 3x)^7$. Without the Chain Rule, we can do this one. All we have to do is expand it, i.e, multiply it 7 times. That will not be fun! Instead, we will use the Chain Rule! We observe that if $g(x) = x^2 + 3x$, then $f\big(g(x)\big) = (x^2 + 3x)^7$.

The Chain Rule is as follows: If $F(x) = f\big(g(x)\big) \rightarrow F'(x) = f'\big(g(x)\big)g'(x)$. Or in Laymen's terms:

DERIVATIVE OF THE OUTSIDE TIMES DERIVATIVE OF THE INSIDE.

We Note that the $g(x)$ is the inside, and the $f(g(x))$ is the outside. Some functions are easier to see this way than others.

Leibnitz Notation: Let $g(x) = u$, and $y = f(x)$. Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Let's go back to our example: If $F(x) = (x^2 + 3x)^7$, and $(x^2 + 3x)$ is the inside, then $F'(x) = 7(x^2 + 3x)^6 \cdot (2x + 3)$, where $7(x^2 + 3x)^6$ is the derivative of the outside, and $(2x + 3)$ is the derivative of the inside.

Let's re-label everything and use Leibnitz notation: Let $u = x^2 + 3x$, and $y = u^7$. Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \rightarrow 7u^6(2x + 3) = 7(x^2 + 3x)^6(2x + 3)$, since $u = x^2 + 3x$. So we get the same answer using both notations (which of course we would).

PROOF: We will only prove this for a particular point, i.e. at $x = a$. The general proof is left for the student to explore at will.

To prove the Chain Rule at $x = a$, we will use $\lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$ notation. Let $F(x) = f\big(g(x)\big)$. Then $F'(x) = \lim\limits_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} = \text{(Warning: Trick!)} = \lim\limits_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \cdot \frac{g(x) - g(a)}{g(x) - g(a)} =$

$$\lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} \cdot \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \text{ (by Limit Law 3)} = f'\big(g(a)\big)g'(a).$$

(Note: trick is in red).

EXAMPLE:

1) Let $f(x) = \sqrt{2x^3 + 9x} = (2x^3 + 9x)^{\frac{1}{2}} \rightarrow\ f'(x) = \frac{1}{2}(2x^3 + 9x)^{-\frac{1}{2}}(6x^2 + 9) = \frac{6x^2+9}{2\sqrt{2x^3+9x}}$, where $\frac{1}{2}(2x^3 + 9x)^{-\frac{1}{2}}$ is the derivative of the outside, and $6x^2 + 9$ is the derivative of the inside.

2) Let $y = (3e^x + \sin x - 2x)^3 \rightarrow \frac{dy}{dx} = 3(3e^x + \sin x - 2x)^2(3e^x + \cos x - 2)$ where $3(3e^x + \sin x - 2x)^2$ is the derivative of the outside, and $3e^x + \cos x - 2$ is the derivative of the inside.

3) Let $f(x) = e^{2x} \rightarrow f'(x) = 2e^{2x}$ (This one can perhaps be more difficult to see. Because of the terminology we have used: (I.e., outside and inside), it may be more difficult to understand which is which). The outside here would be e^u, and the inside would be $u = 2x$. Therefore, the derivative of the outside is e^{2x} and the derivative of the inside is 2. Or: $f'\big(g(x)\big)g'(x) = e^{2x}(\text{which is } f'\big(g(x)\big), \text{times } 2\big(\text{which is } g'(x)\big) = 2e^{2x}$.

4) Let $f(x) = e^{x^2-x} \rightarrow f'(x) = (2x - 1)e^{x^2-x}$ (Note: the exponential, e, piece never changes).

5) Let $y = \sin^2 x = (\sin x)^2 \rightarrow \frac{dy}{dx} = 2\sin x \cos x$ where $2\sin x$ is the derivative of the outside, and $\cos x$ is the derivative of the inside.

6) So far, the above examples are pretty straightforward. This is due to the fact that we only had to apply the Chain Rule by itself (In addition to the more basic rules). We will now combine the Chain Rule with the Product and Quotient Rules.

   Let $f(x) = (3x^2 + 6x)^4(x^3 + 3x^2)^5$. Can you pick out which rules you will have to use? Hopefully you can see it is the Product Rule and the Chain Rule. Observe that we have multiplication, and we have composite functions. The next question will be which rule do I start with? We will start with the Product Rule, because we have a product of two functions, each with a different power. If they were combined in a way to be raised to the same power, we would have started with the Chain Rule.

   $$f'(x) = \underbrace{(3x^2 + 6x)^4}_{1^{st}} \cdot \underbrace{5(x^3 + 3x^2)^4(3x^2 + 6x)}_{\substack{d(2^{nd}) \\ \text{Chain Rule}}} + \underbrace{4(3x^2 + 6x)^3(6x + 6)}_{\substack{d(1^{st}) \\ \text{Chain Rule}}}\underbrace{(x^3 + 3x^2)^5}_{2^{nd}}.$$

7) Next let's try a Quotient Rule/Chain Rule combo: Let $f(x) = \left(\frac{3x^2-9}{x^3+6x}\right)^3$. We will start by using the Chain Rule first. (Notice that it is all to a single power).

$$f'(x) = 3\left(\frac{3x^2-9}{x^3+6x}\right)^2 \left[\frac{(x^3+6x)(6x-9)-(3x^2+6)(3x^2-9)}{(x^3+6x)^2}\right]$$

d(outside) d(inside)

Quotient Rule

8) Let $y = (3x^3+6x^2+7x)^2(5x^2+9x-1)^3 \rightarrow$
$\frac{dy}{dx} = (3x^3+6x^2+7x)^2 \cdot 3(5x^2+9x-1)^2(10x+9) +$
$2(3x^3+6x^2+7x)^2(9x^2+12x+7) \cdot (5x^2+9x-1)^3$.

9) Let $y = \sqrt{\frac{x^2+2x}{2x-9}} = \left(\frac{x^2+2x}{2x-9}\right)^{\frac{1}{2}} \rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{x^2+2x}{2x-9}\right)^{-\frac{1}{2}} \left[\frac{(2x-9(2x+2)-2(x^2+2x)}{(2x-9)^2}\right]$

d(outside) d(inside)

Quotient Rule

10) Let $f(x) = \frac{(x^2-3x)^3}{(e^x-5x^2)^7}$. For this one, we need to start with the Quotient Rule first, since we have two functions to different powers: $f'(x) = \frac{(e^x-5x^2)^7 \cdot 3(x^2-3x)^2(2x-3)-7(e^x-5x^2)^6(e^x-10x)(x^2-3x)^3}{(e^x-5x^2)^{14}}$

11) Let $f(x) = e^x \sin^2 x \rightarrow f'(x) = e^x \cdot 2\sin x \cos x + e^x \sin^2 x$

1st d(2nd) d(1st) 2nd

Chain Rule

12) Let $f(x) = (\tan^2 x - e^{2x})^5 \rightarrow f'(x) = 5(\tan^2 x - e^{2x})^4\,(2\tan x\, sec^2 x - 2e^{2x})$

d(outside) d(inside)

Chain Rule Chain Rule

13) Let $y = \frac{2}{4x^2-9} = 2(4x^2-9)^{-1} \rightarrow \frac{dy}{dx} = -2(4x^2-9)^{-2}8x = -\frac{16x}{(4x^2-9)^2}$

14) Let $y = \frac{1}{\sqrt{\sin x+4x^4}} = (\sin x + 4x^4)^{-\frac{1}{2}} \rightarrow \frac{dy}{dx} = -\frac{1}{2}(\sin x + 4x^4)^{-\frac{3}{2}}(\cos x + 16x^3) =$
$-\frac{\cos x+16x^3}{2\sqrt{(\sin x+4x^4)^3}}$

15) Let $f(x) = \cos(e^x)$. Let us notice that this is a Chain Rule problem, not a Product Rule problem. You have to take the cosine of something (another common mistake). $f'(x) = -\sin(e^x) \cdot e^x$.

16) Let $f(t) = \cos(\cos(e^x))$. This one is a double Chain Rule Problem.

$$f'(t) = -\sin(\cos(e^x)) \cdot (-\sin(e^x)) \cdot e^x$$

17) Let $y = e^{\sin(x^2)} \rightarrow \frac{dy}{dx} = e^{\sin(x^2)} \cdot \cos(x^2) \cdot 2x$ (Another multiple Chain Rule Problem).

18) Find $f''(x)$ for $f(x) = (2x^3 + 10x^2)^3$

$f'(x) = 3(2x^3 + 10x^2)^2(6x^2 + 20x)$

$f''(x) = 3(2x^3 + 10x^2)^2(12x + 20) + 6(2x^3 + 10x^2)(6x^2 + 20x)(6x^2 + 20x)$

1st d(2nd) d(1st) 2nd

Chain Rule

(Note: You get the $(6x^2 + 20x)$ twice. One is the derivative of the inside, and the other is the second function. This frequently happens with 2nd derivatives of this type).

19) Find the equation of the tangent line to the graph $y = (3x^2 - 9)^2$ at the point (2,9).

$f'(x) = 2(3x^2 - 9)6x \rightarrow f'(2) = 72 \rightarrow\ y - 9 = 72(x - 2) \rightarrow y = 72x - 135.$

20) Let $f(x) = a^x$ where a is any constant > 0, and $\neq 1$. We can write $a = e^{\ln a}$. Then $f(x) = a^x = e^{(\ln a)^x} = e^{x \ln a}$ or $e^{(\ln a)x} \rightarrow\ f'(x) = e^{(\ln a)x} \cdot \frac{d(x \ln a)}{dx} = a^x \ln a$, since $a = e^{\ln a}$. (Also note that $\frac{d(x \ln a)}{dx} = \ln a$, because $\ln a$ is a constant, which many students forget).

Let $f(x) = 3^{x^2-9} \rightarrow f'(x) = 3^{x^2-9} \cdot \ln 3 \cdot (2x) = 2 \ln 3 \cdot x \cdot 3^{x^2-9}$

d(outside) d(inside)

EXERCISES:

Differentiate:

1) $f(x) = (5x^3 - 7x^2 + 9)^7$
2) $f(x) = (sinx + x^2 + 4e^x)^3$
3) $y = \left(\frac{1}{2}e^x - 12x^2 - 9x + 2\right)^2$
4) $y = \sqrt{x^{-2} - tanx}$
5) $y = \frac{1}{\cos^2 x - cotx}$
6) $y = \frac{2}{(2x^2 - e^{3x} + secx)^3}$
7) $y = e^{5x^7 - 2x^3 + 4} + \tan^2 x - 4x^5$
8) $f(x) = 3x^2(x^3 + 7x - 2)$
9) $f(x) = (x^2 - 9)^3(x^3 + 2x)^5$
10) $y = (e^x - cosx)^9(4x^7 - 2)^4$
11) $f(x) = (7x^2 + 2x + 8)^2(6x^3 + 2x^2 + x)^4$
12) $f(x) = \sqrt{14x^2 - 2x} \cdot (4x^8 - 10x + 3)^9$
13) $y = (\sin x + \pi x - x^2)^{-3}(3x^2 + 7x)^{\frac{3}{2}}$
14) $y = \left(\frac{12x^2 - 5x}{x - 2}\right)^5$
15) $f(x) = \sqrt{e^{\frac{1}{2}x} + tanx + x^{\frac{5}{3}}}$

16) $y = \left(\frac{e^{4x} - x^{-2} + 7}{x^2 + 6x}\right)^{-2}$

17) $f(x) = \sqrt{\frac{5x^4 - 3x + 2}{x^2 - x}}$

18) $f(x) = \left(\frac{sin^2 x - e^{2x} - cscx}{4x^3 - 12}\right)^{\frac{2}{3}}$

19) $y = \frac{(4x^7 + 3x + 5)^7}{(e^{8x} - 4x + 6)^3}$

20) $y = \frac{(12x^2 - 7)^{\frac{1}{2}}}{(14x^3 - 20x^2 + 6x)^2}$

21) $y = \frac{(tanx + \cos^2 x - x)^2}{(5x^2 - 5e^x)^4}$

22) $f(x) = \sin(sinx)$

23) $f(x) = \cos(\tan^2 x)$

24) $f(x) = e^{\cos^2 x}$

25) $f(t) = \cos(\cos(cost))$

26) $y = \tan^2(e^{10x} + \sin^2 x)$

27) $y = \left(\sin\left(e^{3x^3} + 9x\right)\right)^3$

28) $f(t) = \sqrt{\tan(sin2x - e^{2x})}$

29) $s(t) = e^{\sin^2 x - \cot^2 x}$

30) $y = e^{\sin^2 x + \cos^2 x}$

31) $f(x) = (3^x - e^{4x^2})$

32) $f(x) = 2^{4x^3-2x}$

Find $\frac{d^2y}{dx^2}$ for the following functions:

33) $y = (4x^7 - 3x^2)^5$

34) $y = (sinx + \cos^2 x - x)^{\,2}$

35) $y = (e^x - 4x^2)^7$

36) $y = (e^{3x} + 7x^2 - 9)^{\frac{1}{3}}$

37) $y = \sqrt{3x^3 - 7x}$

Find the equation of the tangent line to the graph at the given point:

38) $f(x) = (3x^2 - 9)^3, \;\; (2,27)$

39) $y = \sin^2 x - e^{2x}, \;\; (0,-1)$

40) $y = \sqrt{x^2 - 4}, \;\; (3,5)$

41) $f(x) = \sin(\sin x)\,, \;\; (0,0)$

CHAPTER 2
SECTION 5

DIFFERENTIATION
IMPLICIT DIFFERENTIATION

Let us start by doing a few familiar things: Let $f(x) = x^2 \rightarrow f'(x) = 2x$. Let $y = \sin x \rightarrow \frac{dy}{dx} = \cos x$. Let $f(x) = (x^2 + \sin x)^3 \rightarrow f'(x) = 3(x^2 + \sin x)^2(2x + \cos x)$.

These should all look familiar to you by now. What do they all share in common? You may not be able to guess where we are headed. The property they all have in common is that they are all functions of x. You can write each one as $y = f(x)$. But what if we wanted to find the derivative of something like $x^2 + y^2 = 1$ (the unit circle, which is not a function). You could solve for y first, but then you have this $\pm$ case, which is not all that convenient. Now, what if we had something like $y^2 + 3xy^2 - 2x^2y = 3y^3$? This is even more difficult, because we don't know how to solve for y. We will not need to. Instead, we will employ Implicit Differentiation.

Recall the Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. We will use this to differentiate equations with respect to x, (equations that are not explicit functions of x. Instead, we will use the fact that they are **implicit** functions of x, hence the term Implicit Differentiation).

Steps to find $\frac{dy}{dx}$ implicitly: (Note: Some texts use y' interchangeable with $\frac{dy}{dx}$. I will not do this, as I prefer Leibnitz' notation $\frac{dy}{dx}$ as it is less vague). (Also Note: These steps are for finding $\frac{dy}{dx}$ only, meaning we are considering y as an implicit function of x, i.e. we are taking the derivative with respect to x.

1) Take derivatives of **both** sides: (Some students forget to take the derivative of the RHS).
   a) When taking a derivative involving x, **do it the same as we always have**.
   b) When taking the derivative involving y, **multiply that term by $\frac{dy}{dx}$**. (This is the Chain Rule part. This is because y is implicitly a function of x, and we are taking the derivative with respect to x. We could multiply the derivative involving x by $\frac{dx}{dx}$, but since this is just 1, there is no need). (Also note that we have already been doing this without knowing it. E.G., if $y = x^2 \rightarrow \frac{dy}{dx} = 2x \rightarrow \frac{d(y)}{dx} = 1 \cdot \frac{dy}{dx} = \frac{dy}{dx}$.)
2) Solve for $\frac{dy}{dx}$:
   a) Get all the terms with $\frac{dy}{dx}$ on one side, and all the terms without $\frac{dy}{dx}$ on the other side.
   b) Factor out the $\frac{dy}{dx}$.
   c) Divide both sides of the equation to isolate $\frac{dy}{dx}$ (if necessary).

Let's go back to our 2 original examples:

$\boldsymbol{x^2 + y^2 = 1}$:

1) Take derivatives of both sides:

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

Chain Rule

2) Solve for $\frac{dy}{dx}$:

$$2y \cdot \frac{dy}{dx} = -2x:$$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

$\boldsymbol{y^2 + 3xy^2 - 2x^2y = 3y^3}$:

This one will involve the Product Rule as well:

1) Take derivatives of both sides:

$$2y \cdot \frac{dy}{dx} + 3x \cdot 2y \cdot \frac{dy}{dx} + 3y^2 - 2x^2 \cdot \frac{dy}{dx} - 4xy = 9y^2 \cdot \frac{dy}{dx}:$$

Product Rule Product Rule

2) Solve for $\frac{dy}{dx}$:

$$2y \cdot \frac{dy}{dx} + 6xy \cdot \frac{dy}{dx} - 2x^2 \cdot \frac{dy}{dx} - 9y^2 \cdot \frac{dy}{dx} = -3y^2 + 4xy:$$

Factor out the $\frac{dy}{dx}$ to get: $\frac{dy}{dx}(2y + 6xy - 2x^2 - 9y^2) = -3y^2 + 4xy$:

Divide: $\frac{dy}{dx} = \frac{-3y^2+4xy}{2y+6xy-2x^2-9y^2}$

EXAMPLE:

1) $2y^4 - 2xy = 7x$: Find $\frac{dy}{dx}$:

$$8y^3 - 2x \cdot \frac{dy}{dx} - 2y = 7$$

Product Rule

$$-2x \cdot \frac{dy}{dx} = 7 - 8y^3 + 2y \rightarrow \quad \frac{dy}{dx} = \frac{-8y^3 + 2y + 7}{-2x} = \frac{8y^3 - 2y - 7}{2x}.$$

2) $\sin^2 x - e^{xy} = 4y^{-3}$. Find $\frac{dy}{dx}$:

$$2\sin x \cos x - e^{xy}\left(x \cdot \frac{dy}{dx} + y\right) = -12y^{-4} \cdot \frac{dy}{dx}$$

Chain/Product Rules

$$2\sin x \cos x - xe^{xy} \cdot \frac{dy}{dx} - ye^{xy} = -12y^{-4} \cdot \frac{dy}{dx}$$

$$-xe^{xy} \cdot \frac{dy}{dx} + 12y^{-4} \cdot \frac{dy}{dx} = -2\sin x \cos x + ye^{xy}$$

$$\frac{dy}{dx}(\ -xe^{xy} + 12y^{-4}) = -2\sin x \cos x + ye^{xy}$$

$$\frac{dy}{dx} = \frac{-2\sin x \cos x + ye^{xy}}{-xe^{xy} + 12y^{-4}} = \frac{2\sin x \cos x - ye^{xy}}{xe^{xy} - 12y^{-4}}$$

3) $\sqrt{\cos y + xy} = 3y^2$ Find $\frac{dy}{dx}$:

$$\frac{1}{2}(\cos y + xy)^{-\frac{1}{2}}\left(\sin y \cdot \frac{dy}{dx} + x \cdot \frac{dy}{dx} + y\right) = 6y \cdot \frac{dy}{dx}$$

$$\frac{\sin y}{2\sqrt{\cos y + xy}} \cdot \frac{dy}{dx} + \frac{x}{2\sqrt{\cos y + xy}} \cdot \frac{dy}{dx} + \frac{y}{2\sqrt{\cos y + xy}} = 6y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{\sin y}{2\sqrt{\cos y + xy}} + \frac{x}{2\sqrt{\cos y + xy}} - 6y\right) == -\frac{y}{2\sqrt{\cos y + xy}}$$

$$\frac{dy}{dx}=\frac{-\frac{y}{2\sqrt{\cos y+xy}}}{\frac{\sin y}{2\sqrt{\cos y+xy}}+\frac{x}{2\sqrt{\cos y+xy}}-6y}=\frac{-y}{\sin y+x-12y\sqrt{\cos y+xy}}$$

4) Let us now find $\frac{dy}{dx}$ at a particular point. We will proceed as we have before. We can substitute our point in for (x, y) at the end, or do it after we take the derivative. Let's go ahead and do it before solving for $\frac{dy}{dx}$:

Find $\frac{dy}{dx}$ for $3x^2y-2x^2=5y$ at the point (1,-1):

$$3x^2\cdot\frac{dy}{dx}+6xy-4x=5\cdot\frac{dy}{dx}$$

Now let's substitute (1,-1) for (x, y): $3\cdot1^2\cdot\frac{dy}{dx}-6\cdot1\cdot(-1)-4\cdot1=5\cdot\frac{dy}{dx}\rightarrow$
$3\cdot\frac{dy}{dx}+6-4=5\cdot\frac{dy}{dx}\rightarrow\quad 2\cdot\frac{dy}{dx}=2\rightarrow\quad\frac{dy}{dx}=1$

5) Let us find the equation of the tangent line for the equation in Example 4): We found $\frac{dy}{dx}=1$ at the point (1,-1). Therefore, $y+1=x-1\rightarrow\quad y=x-2$.

6) Let's find the equation of the tangent line for the circle $x^2+y^2=4$ at the point (0,2): First we find $\frac{dy}{dx}$: $2x+2y\cdot\frac{dy}{dx}=0\rightarrow\quad\frac{dy}{dx}=-\frac{x}{y}$. Substituting in (0,2) for (x, y) we get $-\frac{0}{2}=0$. Therefore, $y-2=0\rightarrow\quad y=2$. (Note: we cannot find $\frac{dy}{dx}$ at (2,0) or at (-2,0), because the tangent line would be vertical with slopes equal to 0. We recall an expression is not differentiable where it has a vertical tangent).

7) We will now do something a little different. We will find $\frac{dx}{dy}$ for an expression. How will we do this? Now, when taking the derivative involving an x we will multiply that term by $\frac{dx}{dy}$ and when we take the derivative involving a y we won't multiply it by $\frac{dy}{dx}$, because in this case we are considering x to be a function of y.

$2xy+y^2-3x=10$:

$$2x+2\cdot\frac{dx}{dy}\cdot y-3\cdot\frac{dx}{dy}=0$$

$$2y\cdot\frac{dx}{dy}-3\cdot\frac{dx}{dy}=-2x$$

$$\frac{dx}{dy}(2y-3) = -2x$$

$$\frac{dx}{dy} = -\frac{2x}{2y-3}$$

EXERCISES:

Find $\frac{dy}{dx}$ by using Implicit Differentiation:

1) $x^2 + y^2 = 9$

2) $3x^2 - 2y^2 = 7$

3) $4x^2 - 2y = 3x$

4) $2xy^2 - 3x^2 = 4y$

5) $3x^3y^2 - 4xy = 2y^3$

6) $4x^2 - 2xy - 12x = xy$

7) $4ye^x - \sin x = \tan^2 y$

8) $\sqrt{x^3 - \sin^2 x} = e^y$

9) $e^{2xy} - \cos y = \sin(xy)$

10) $\cot(xy) = \sqrt{3x^2y}$

11) $(4xy - \tan(2xy) + e^y)^= 1$

12) $\sin(x - 2y) = 4e^{2x} - xy^{-2}$

13) $\tan(2xy^2) + 4e^{x^2y} = 4y^7$

Find $\frac{dy}{dx}$ at given points:

14) $x^2 + 2y^2 = 9$ at the point (1,2)

15) $\sin(xy) - e^x = \frac{1}{2}y$ at the point (0,-2)

16) $4xy + 2x = 4y$ at the point (2,-1)

17) Find the equation of the tangent line to the graphs in Exercises 14)-16) at the given points in Exercises 14)-16).

18) Find $\frac{dx}{dy}$ for the equations in Exercises 1) through 4).

CHAPTER 2
SECTION 6

DIFFERENTIATION
RELATED RATES

What is a Related Rate? First, it is an application of Implicit Differentiation, which we learned in the last section. Second, it is a way of calculating a rate based on another known rate, related to the rate of interest.

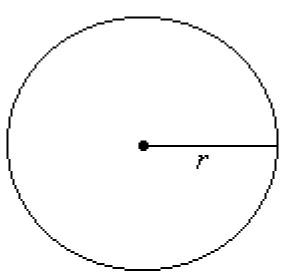
For Example: Let's say we have a circular object (perhaps a perfect circular puddle?), and it is growing. Let us also assume that we know the rate at which the radius is growing. By knowing that information, and how to perform Implicit Differentiation, we can calculate how fast its area is growing.

STEPS FOR CALCULATING A RELATED RATE:

1) Draw a picture (or diagram) if possible.
2) Label the picture and/or choose variables (if no picture is useful).
3) Write an equation with the variables chosen that relates the information.
4) Take time derivatives of both sides using Implicit Differentiation (Chain Rule). What does this mean? You must take derivatives of each term, and you must multiply each one by $\frac{d(?)}{dt}$ where the ? is whatever variable you took the derivative of. This is isomorphic to when we multiplied the term by $\frac{dy}{dx}$ anytime we took the derivative involving a y. We are now taking the derivative with respect to t. In this case none of the variables in our equation will be t (for time), so we have to multiply by $\frac{d(?)}{dt}$ in each case.
5) Substitute in all know values. (Note: sometimes students do this step too soon, and get $0 = 0$).
6) Solve for the unknown rate desired.

EXAMPLE: Let's go back to our puddle example. Let's say we have a completely circular puddle that has the radius increasing at a rate of 2 inches per second. At what rate is its area increasing, when it's radius is 2 feet?

First we will draw a picture: Then we will label it:



Next, we write an equation: Which equation will we need? The area of a circle:

$A = \pi r^2$

Taking time derivatives of both sides, using Implicit Differentiation:

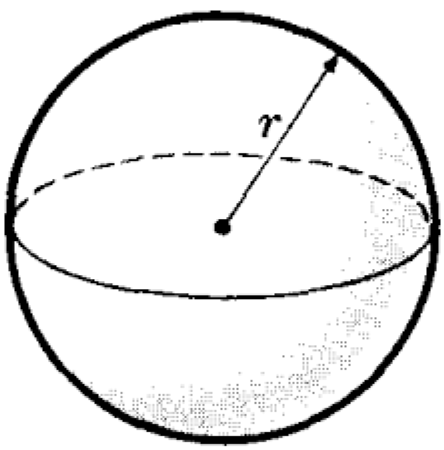
$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt} \rightarrow \ \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

Next, we substitute in all known values:

$$\frac{dA}{dt} = 2\pi \cdot 24 \cdot 2 = 96\pi \frac{inches^2}{second}$$

EXAMPLE: Let us make a snowball, (a perfectly spherical one), that will increase in size as we make it. Its volume is increasing at a rate of $3\frac{cm^3}{s}$. At what rate is its radius increasing when it is $8\ cm$?

Let's draw a picture, and label it:



We need the equation for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Let's take time derivatives of both sides:

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot \frac{3r^2 dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$ (Note that $4\pi r^2$ is the Surface Area of a sphere! Pretty Cool right?!)

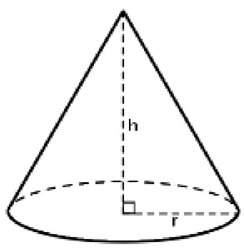
Next, we substitute in all known values:

$$3 = 4\pi \cdot 8^2 \cdot \frac{dr}{dt}$$

Lastly we solve for $\frac{dr}{dt}$:

$$\frac{dr}{dt} = \frac{3}{256\pi}\frac{cm}{s}$$

EXAMPLE: Sand is being dumped by a dump truck forming a pile of sand in the shape of a cone. The radius is increasing at a rate of 4 m/s, and the height is increasing at a rate of 6 m/s. How fast is the volume increasing when the radius is 10 meters, and the height is 15 meters?



Next, we need an equation for the volume of a cone:

$$V = \frac{1}{3}\pi r^2 h$$

Taking time derivatives of both sides: (We note that we have 3 related rates here instead of two as in the previous examples).

$\frac{dV}{dt} = \frac{2}{3}\pi r \cdot \frac{dr}{dt} \cdot h + \frac{1}{3}\pi r^2 \cdot \frac{dh}{dt}$ using the Product Rule.

Substituting the knowns values, we get:

$$\frac{dV}{dt} = \frac{2}{3}\pi \cdot 10 \cdot 4 \cdot 15 + \frac{1}{3}\pi \cdot 10^2 \cdot 6 = 400\pi + 200\pi = 600\pi \frac{m^3}{s}$$

EXAMPLE: Let's explore a business example:

Suppose that price p, in dollars, and number of sales, x, of a certain watch follows the equation $2p + 3x + 4px = 30$. Suppose also that p and x are both functions of time, measured in days. Find the rate at which x is changing when $x = 5, p = 10$, and $\frac{dp}{dt} = 2$:

We already have an equation in this example:
$2p + 3x + 4px = 30$

Let's take time derivatives:

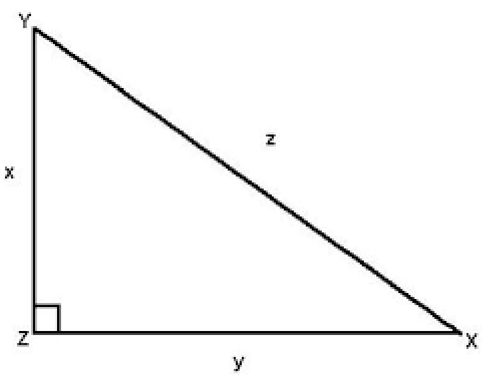
$$2 \cdot \frac{dp}{dt} + 3 \cdot \frac{dx}{dt} + 4p \cdot \frac{dx}{dt} + 4x \cdot \frac{dp}{dt} = 0$$

Now, we will substitute our known values:

$$2 \cdot 2 + 3 \cdot \frac{dx}{dt} + 4 \cdot 10 \cdot \frac{dx}{dt} + 4 \cdot 5 \cdot 2 = 0$$

$43 \cdot \frac{dx}{dt} = -44 \rightarrow \ \frac{dx}{dt} = -\frac{44}{43}\ \frac{\#\, sales}{day}$. (This is negative. That is okay. We are losing some sales per day).

EXAMPLE: A blue car is traveling West at 60 mph, and a red car is traveling South at 70 mph. Both are headed to the same place. At what rate are the cars approaching each other when the blue car is 3 miles from the destination, and the red car is 4 miles from the destination?



We let Z be the destination for both cars. Let y be the distance from the blue car to Z. Let x be the distance from the red car to Z. We need a formula. We use the Pythagorean Theorem:

$$x^2 + y^2 = z^2$$

We note that $\frac{dx}{dt} = -70$ mph, and $\frac{dy}{dt} = -60$ mph. Why are they negative? It is because their distances are decreasing.

Let's take time derivatives:

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2z \cdot \frac{dz}{dt}$$

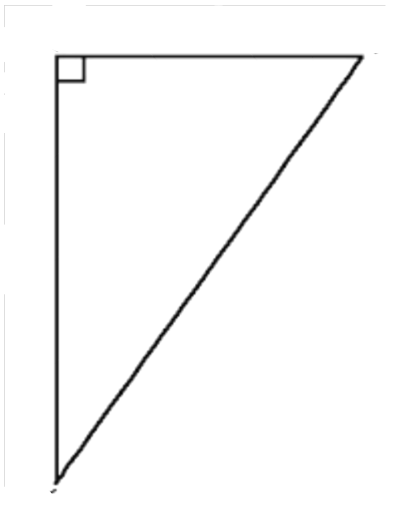
Next, we need to calculate z:

When $x = 4, y = 3$, we get: $4^2 + 3^2 = z^2 \rightarrow \quad z = 5$

Now we substitute all known values:

$2 \cdot 4 \cdot (-70) + 2 \cdot 6 \cdot (-60) = 2 \cdot 5 \cdot \frac{dz}{dt} \rightarrow \qquad \frac{dz}{dt} = -128 \text{ mph}$

EXAMPLE: A plane is flying at an altitude 5.5 miles. It will pass over the Empire State Building.



Let $x =$ horizontal distance, $y =$ vertical distance, and $z =$ hypotenuse. If z is decreasing at a rate of 550 mph when z is 20 miles, what is the speed of the plane? $y = 5.5$ miles. $z = 20$ miles. Let us find x. $x^2 + 5.5^2 = 20^2 \rightarrow \quad x = \sqrt{400 - 30.25} \approx 19.23$.

Our equation is: $x^2 + 5.5^2 = z^2$

Taking time derivatives:

$$2x \cdot \frac{dx}{dt} + 0 = 2z \cdot \frac{dz}{dt}$$

Substituting all knowns:

$$2 \cdot 19.23 \cdot \frac{dx}{dt} = 2 \cdot 20 \cdot 550 \rightarrow \frac{dx}{dt} = 572 \; mph$$

EXERCISES:

1) Let $y = 2x^2y - 3y$. Find $\frac{dy}{dt}$ when $\frac{dx}{dt} = 2$ and $x = 1, y = 0$

2) Let $3z^3 = e^{x^2} + 5y + 18$. Find $\frac{dx}{dt}$ when $\frac{dy}{dt} = 2, \frac{dz}{dt} = 4, x = 0, y = 1, z = 8$

3) Let $e^x = xy^2 - cosz + 2x$. Find $\frac{dz}{dt}$ when $\frac{dx}{dt} = 0, \frac{dy}{dt} = 1, x = 0, y = 3, z = \frac{\pi}{2}$

4) Suppose that price p, in dollars, and number of sales, x, of a certain watch follows the equation $4p + 2x + 3px = 60$. Suppose also that p and x are both functions of time, measured in days. Find the rate at which x is changing when $x = 20, p = 5$, and $\frac{dp}{dt} = 5$:

5) A raindrop keeps growing larger as it spreads out on the concrete. It is growing uniformly, and is perfectly circular. What is the rate of increase of its area, when its radius is $2mm$? Its radius is increasing at a rate of $1\frac{mm}{s}$.

6) A circular water drop is evaporating. Its area is decreasing at a rate of $-3\frac{nm^2}{s}$. At what rate is its radius decreasing , when its radius is $5\ mm$?

7) Sally is making a snowball. It is perfectly spherical. Its volume is growing at a rate of $12\frac{cm^3}{s}$. At what rate is its radius increasing when its radius is $20\ cm$?

8) A snowball is melting. Its radius is shrinking at a rate of $-\frac{2mm}{s}$. At what rate is its Surface Area decreasing when its radius is $8\ cm$?

9) Two cars are headed away from each other. Car A is headed North, and car B is headed East. Car A is traveling at 55 mph, and car B is traveling at 75 mph. At what rate is their distance increasing two hours after they leave each other?

10) A particle travels along the curve $y = 2x + e^z$. The rate in the x-direction is $2\frac{cm}{s}$, and in the z-direction, it travels at $-1\frac{cm}{s}$. At what rate does it travel in the y-direction, when $x = 1, z = 0$?

11) A rectangular ice cube is melting. Its height is shrinking at a rate of $4\frac{mm}{s}$. Its width is shrinking at a rate of $5\frac{mm}{s}$, and its volume is decreasing at a rate of $9\frac{mm^3}{.s}$. At what rate is depth shrinking, when its height is $6\ cm$, its width is $3\ cm$, and its depth is $1.5\ cm$?

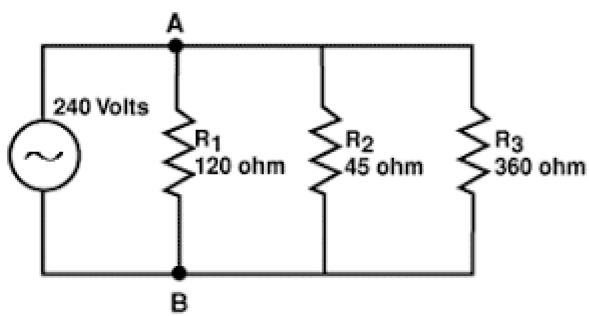
12) A man is running diagonally across a football field. Its dimensions are 360×160 feet. If he is running at 20 $\frac{ft}{s}$, at what rate is his horizontal distance (the 160 ft side) decreasing when he is 20 feet from the corner he is headed to?

13) Coal is being dumped into a pile of in the shape of a cone. The radius is increasing at a rate of 7 m/s, and the height is increasing at a rate of 10 m/s. How fast is the surface area increasing when the radius is 5 meters, and the height is 10 meters?

14) Liquid is being poured into an object shaped like an upside down cone. Its volume is increasing at a rate of 5 $\frac{cm^3}{s}$, and its radius is increasing at a rate of 2 $\frac{cm}{s}$ At what rate is its height increasing when its height is $12\ cm$, and its radius is $3\ cm$?

15) A flat raft is moving away from a dock. The dock is 2 m higher than the raft. If the raft is moving at a rate of 3 $\frac{m}{s}$, what is the rate it is moving away from the dock, when it is 6 m away?

16)



We have 3 resistors in parallel with resistances as shown. The total resistance, R, is given by the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. If R_1 is increasing by $.1\ \frac{\Omega}{s}$, R_2 is increasing by $.2\ \frac{\Omega}{s}$, and R_3 is increasing by $.4\ \frac{\Omega}{s}$, at what rate is R changing, when R_1, R_2, R_3 are as given in the diagram?

17) Two people are heading toward each other. If one is walking south at 3 mph, and the other is running at 7 mph. At what rate are the two approaching each other when the runner is 5 miles from the destination, and the walker is 12 miles from the destination?

18) A ladder is propped up on a wall. If the top of the ladder slides down at 2 $\frac{ft}{s}$, and the ladder is $6\ ft$ tall, how fast is the bottom of the ladder sliding away from the wall when it is $5\ ft$ from the wall?

19) A plane is flying at an altitude 6.5 miles. It will pass over a radar station. If its rate from the plane to the station is decreasing at a rate of 550 mph when z is 20 miles, what is the speed of the plane?

20) A right triangle has sides, 7, 24, 25 cm. If the width of the triangle is growing at a rate of $1\frac{cm}{s}$, and the height is growing at a rate of $2\frac{cm}{s}$, at what rate is the hypotenuse increasing?

21) A right triangle has its length decreasing at a rate of $1\ \frac{\text{cm}}{s}$. How fast is the angle decreasing when its length is 4 and its width is 3 if the base is fixed?.

22) A cylinder is filled with oil. If its volume is increasing at a rate of $4\frac{cm^3}{s}$, at what rate is its height increasing when its height is 8 cm, and its radius is 2 cm?

CHAPTER 2
SECTION 7

DIFFERENTIATION
DERIVATIVES OF LOGARITHMIC FUNCTIONS AND INVERSE TRIGONOMETRIC FUNCTIONS

THE NATURAL LOGARITHMIC FUNCTION:

Let's begin by using base e for our logarithmic functions.

Let us recall a logarithmic function and what it is: Recall that it is simply the inverse of the exponential function. We start with $f(x) = a^x$, as a one-to-one function. This gives $y = a^x$. To derive its inverse: $x = a^y$. This is the logarithmic function. With functions, we want to solve for y. So to solve for y, we get the notation $y = \log_a x$. Using base e, we get $f(x) = \ln x$.

Next, we derive the derivative:

This time we will not use the definition, but instead use the rules we have learned thus far:

First: $f(x) = \ln x \rightarrow \quad y = \ln x$

Rewriting as an exponential:

$e^y = x$

We will next take derivatives of both sides using Implicit Differentiation:

$$e^y \cdot \frac{dy}{dx} = 1 \rightarrow$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

But recall $e^y = x$

Therefore $\frac{dy}{dx} = \frac{1}{x}$.

To summarize: If $\boldsymbol{f(x) = \ln x} \rightarrow \quad \boldsymbol{f'(x) = \frac{1}{x}}$.

This is the base we will use most often.

For other bases:

$y = \log_a x$

Rewrite as an exponential:

$$a^y = x \rightarrow \qquad a^y \cdot \ln a \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \cdot \ln a}$$

CHAIN RULE:

Recall the Chain Rule: Let $F(x) = f(g(x))$. Then $F'(x) = f'(g(x))g'(x)$. We also said, the derivative of the outside times the derivative of the inside.

How does this apply to the Natural Logarithmic Function? Let $f(x) = \ln(u)$, and let $u = g(x)$. Then $f'(g(x))g'(x) = \frac{1}{\text{u}} \cdot u' = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$. (Note that the idea of the derivative of the outside times the derivative of the inside still applies here, where $\ln(u) = \ln(g(x))$ is the outside function, and $u = g(x)$ is the inside function.

EXAMPLE:

1) Let $f(x) = \ln(x^2 + 1)$. Find $f'(x)$. $f'(x) = \frac{1}{x^2+1} \cdot (2x)$, where $\frac{1}{x^2+1}$ is the derivative of the outside, and $2x$ is the derivative of the inside.

2) Let $y = e^x \ln(sinx + 3x^3)$. Find $\frac{dy}{dx}$: $\frac{dy}{dx} = e^x \cdot \frac{1}{sinx+3x^3} \cdot (cosx + 9x^2) + e^x \ln(\sin x + 3x^3) = \frac{e^x cosx + 9x^2 e^x}{sinx+3x^3} + e^x \ln(\sin x + 3x^3)$. We used the Product Rule with the Chain Rule, starting with the Product Rule.

3) Let $y = \ln(\ln(\ln(x)))$. Find $\frac{dy}{dx}$. First, we note that this is a multiple Chain Rule Problem. We get: $\frac{dy}{dx} = \frac{1}{\ln(ln(x))} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}$.
$d(most\ outside) \cdot d(next\ outside) \cdot d(inside)$ (not very formal, but will give you the idea in simple language).

RULES OF LOGARITHMS:

Let us briefly review the rules for Logarithms that we recall from our algebra course:

1) $\log_a xy = \log_a x + \log_a y$

2) $\log_a \frac{x}{y} = \log_a x - \log_a y$

3) $\log_a x^c = c \cdot \log_a x$

(Note: We will not prove these here as we have already done so in our algebra courses).

EXAMPLE:

1) Let $f(x) = \ln[(e^x + 7x)\sqrt{x^2 - 3x}]$. This looks challenging at first, but with the use of the Rules for Logarithms, it is much easier. $f(x) = \ln(e^x + 7x) + \frac{1}{2}\ln(x^2 - 3x)$, applying Rule 1 and Rule 3. Then $f'(x) = \frac{e^x+7}{e^x+7x} + \frac{2x-3}{2(x^2-3x)}$

2) Let $f(x) = ln\left(\frac{x^4+3x}{x-1}\right)$. Again, we will use Rules for Logarithms. Using Rule 2:

   $f(x) = \ln(x^4 + 3x) - \ln(x - 1)$. Then we get: $f'(x) = \frac{4x^3+3}{x^4+3x} - \frac{1}{x-1}$.

3) Let $y = ln\left[\frac{(\sin x+2x^5)^3}{x^2-10x}\right] = 3\ln(\sin x + 2x^5) - \ln(x^2 - 10x)$. Then $\frac{dy}{dx} = \frac{3\cos x+30x^4}{\sin x+2x^5} - \frac{2x-10}{x^2-10x}$.

LOGARITHMIC DIFFERENTIATION:

What is Logarithmic Differentiation? It is a way to make a complicated derivative much easier by taking the Natural Logarithm of both sides, applying Rules of Logarithms, and then taking the derivative of both sides using Implicit Differentiation.

STEPS FOR LOGARITHMIC DIFFERENTIATION:

1) Take the Natural Logarithm of both sides of the equation.

2) Use Rules of Logarithms to fully expand the Logarithms.

3) Take the derivative of both sides with respect to x, using Implicit Differentiation.

4) Solve for $\frac{dy}{dx}$.

EXAMPLE: Let $y = \frac{(3x^2-9)^{\frac{1}{3}}(e^x-2x)}{(4x^2-9)^2(6x-2)}$. In observing this function, we note that we would have to use the Chain Rule, the Quotient Rule, and the Product Rule all at the same time. Let us use Logarithmic Differentiation instead.

First, take the ln of both sides: $\ln y = ln\left[\frac{(3x^2-9)^{\frac{1}{3}}(e^x-2x)}{(4x^2-9)^2(6x-2)}\right]$. Next, we will use Rules of Logarithms to rewrite the Right Hand Side. $\ln y = \frac{1}{3}\ln(3x^2-9) + \ln(e^x - 2x) - 2\ln(4x^2-9) - \ln(6x-2)$. It already looks better!

Next, we take the derivative of both sides, with respect to x, using Implicit Differentiation: $\frac{1}{y}\cdot\frac{dy}{dx} = \frac{6x}{9x^2-27} + \frac{e^x-2}{e^x-2x} - \frac{16x}{4x^2-9} - \frac{6}{6x-2}$. (Not bad compared to what it would have been!). Then Solving for $\frac{dy}{dx}$: $\frac{dy}{dx} = y\left(\frac{6x}{9x^2-27} + \frac{e^x-2}{e^x-2x} - \frac{16x}{4x^2-9} - \frac{6}{6x-2}\right) = \frac{(3x^2-9)^{\frac{1}{3}}(e^x-2x)}{(4x^2-9)^2(6x-2)}\left(\frac{6x}{9x^2-27} + \frac{e^x-2}{e^x-2x} - \frac{16x}{4x^2-9} - \frac{6}{6x-2}\right)$.

POWER RULE PROOF:

We finally prove the Power Rule. We can now do this proof without the definition (which we skipped previously, due to its complexity), by using rules that we have recently learned, including the derivative of the Logarithmic Function, Implicit Differentiation, and Logarithmic Differentiation.

Recall the Power Rule: If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$.

PROOF: First, we take the Natural Logarithm of both sides: $\ln y = \ln(x^n)$. Next, we apply Rule 3 of Logarithmic Rules to get: $\ln y = n \ln x$. Now, we take the derivative of both sides with respect to x, using Implicit Differentiation: $\frac{1}{y}\cdot\frac{dy}{dx} = \frac{n}{x} \rightarrow \quad \frac{dy}{dx} = y\cdot\frac{n}{x}$. But $y = x^n$. So, therefore: $\frac{dy}{dx} = x^n\cdot\frac{n}{x} = nx^{n-1}$. ($x \neq 0$.)

__**DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS:**__

Let's start with the Inverse Sine Function or $f(x) = \sin^{-1} x = \arcsin x$. So $x = \sin y$ when $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

(Recall this is how we write the inverse function). Then taking derivatives of both sides with respect to x: $1 = \cos y\cdot\frac{dy}{dx} \rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y}$. Now, we can write $\cos y = \sqrt{1-\sin^2 y}$ (by using the Pythagorean Trigonometric Identity). But recall we have $x = \sin y$, so $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

So the derivative of $\mathbf{\sin^{-1} x = \arcsin x} = \frac{\mathbf{1}}{\sqrt{\mathbf{1-x^2}}}$.

The Derivatives of all of the Inverse Trigonometric Functions are as follows: (Note the proofs are similar, and left as an exercise for the student):

Function	Derivative
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\csc^{-1} x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$

EXAMPLE:

1) Let $f(x) = 2x \cdot \arcsin x$. Find $f'(x)$: We use the Product Rule: $f'(x) = \frac{2x}{\sqrt{1-x^2}} + 2\arcsin x$.

2) Let $y = \tan^{-1} 2x$. Find $\frac{dy}{dx}$: Here we use the Chain Rule. How do we do this? We substitute the $2x$ for x in the formula for the derivative of $y = \tan^{-1} x$. Then we have to multiply the new function by the derivative of the "inside", which in this case, its derivative is 2. Therefore, $\frac{dy}{dx} =$

   $\frac{1}{1+(2x)^2} \cdot 2 = \frac{2}{1+4x^2}$.

3) Let $f(x) = \arccos x^3$. Then $f'(x) = -\frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2 = -\frac{3x^2}{\sqrt{1-x^6}}$.

4) Let $y = \cot^{-1}(x^2+2)$. Find, $\frac{dy}{dx}$: $\frac{dy}{dx} = -\frac{1}{1+(x^2+2)^2} \cdot 2x = -\frac{2x}{x^4+4x^2+5}$.

EXERCISES:

Differentiate:

1) $f(x) = \ln x^2$

2) $f(x) = \ln(2x - 4)$

3) $y = 3x^2 \ln(3x)$

4) $y = e^x \ln x$

5) $y = \ln(4x^3 - x^{-2})$

6) $f(x) = \frac{\ln(x^2-9)}{e^x+6x^3}$

7) $y = \ln(e^x)$

8) $y = \ln[(x^2 - 9)(4x^3 + 2x)]$

9) $f(t) = ln[(2x + 7)^2(3x^4 + 3x^2)^3]$

10) $y = \ln\left[(2x - 2)^3\sqrt{x^2 + 9x}\right]$

11) $f(x) = ln\left[\frac{x+1}{2x-2}\right]$

12) $f(x) = ln\left[\frac{(14x^3+2x^2-9x)^3}{(sinx-12x)^4}\right]$

13) $f(x) = ln\left[\frac{\sqrt{2x+3}}{(4x^2-3x)^2}\right]$

14) $y = ln\sqrt{\frac{cosx+x^{\frac{1}{3}}}{3x-9}}$

15) $y = \ln(\ln(\ln(x^2)))$

16) $y = \ln(\ln(\sin x))$

17) $y = ln\sqrt{(\tan^2 x - 5x)}$

18) $y = \arcsin(5x)$

19) $y = \cos^{-1}(3x^2 - 2)$

20) $f(x) = e^{x^2} \tan^{-1} 3x$

21) $f(t) = \frac{\operatorname{arccot}(2x^2)}{\ln(3x)}$

22) $y = \frac{\csc^{-1}(2x-9)}{e^{2x}}$

23) $y = \arcsin(14x^2 - 12x)^{\frac{1}{3}}$

24) $f(x) = arccos\left(\frac{12x^3-9x}{3-x^2}\right)$

25) $f(x) = \arcsin(\arcsin x)$

Find the equation of the tangent line of the following function at the given point:

26) $f(x) = \ln(x^2 - 3)$, $(2{,}0)$:

Find $\frac{dy}{dx}$ for the following functions using Implicit Differentiation:

27) $\ln y = \arctan 2x + y^2 \ln x$

28) $3xy = \cos^{-1} xy$

Use Logarithmic Differentiation to find the derivative of the following functions:

29) $y = \sqrt{x^2 - \ln(3x)}\,(4x^3 + 4)^3$

30) $f(x) = \left(e^{x^2} - x^{-4}\right)^{\frac{1}{3}}(19x - 3)^7$

31) $f(x) = \frac{(x^4-20x)^2}{\sqrt{4x+10}}$

32) $y = x^x$

33) $y = \frac{\left(3x^2-7x\right)^2(4x-9)^3}{(20x^3+2)^4\sqrt{14x^2-8}}$

34) $f(t) = \sqrt{\frac{4x^2-2}{\arcsin x}}$

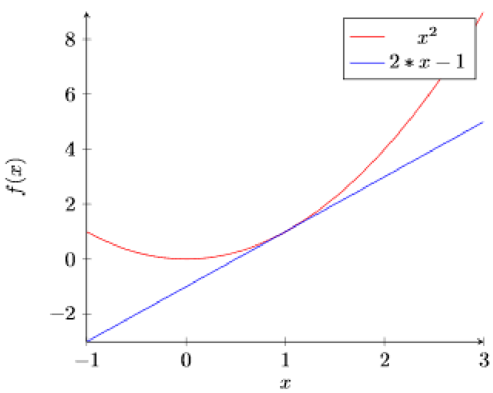
35) $y = x^{e^x}$

Challenge problem: Prove the rest of the Inverse Trigonometric derivatives.

CHAPTER 2
SECTION 8
LINEAR APPROXIMATIONS AND DIFFERENTIALS
HYPERBOLIC FUNCTIONS

Notice when you zoom into a point on the curve of a function, the tangent line approximates the actual curve as you zoom more and more.

Recall the following graph:



The tangent line is plotted at $x = 1$.

Notice, the closer you get to $x = 1$, the more the tangent line looks like the function $f(x) = x^2$.

The equation of the tangent line at $x = a$, is: $y - f(a) = f'(a)(x - a) \rightarrow$

$f(x) = f(a) + f'(a)(x - a)$.

So the Linear Approximation (or tangent line approximation of f at a is:

$\boldsymbol{L(x) = f(a) + f'(a)(x - a)}$.

EXAMPLE:

1) Let $f(x) = x^2$ at $x = 1$ (as in the example above). Then $f'(x) = 2x \rightarrow \quad f'(1) = 2$. The Linear Approximation is $L(x) = 1 + 2(x - 1) \rightarrow \quad L(x) = 2x - 1$, which we observe is the equation of the tangent line that we had above.
2) Let $f(x) = \sqrt{x}$ at $a = 4$. Find the Linear Approximation: $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. At $a = 4$:

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}. \text{ Then } L(x) = 2 + \frac{1}{4}(x - 4) \rightarrow \quad L(x) = \frac{1}{4}x + 1.$$

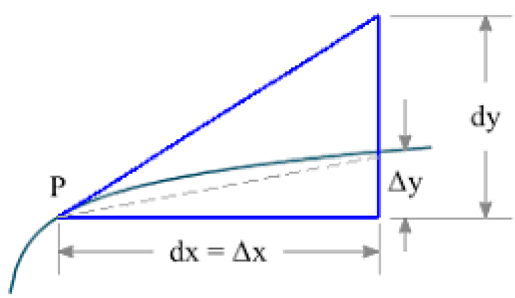
We notice that so far it's no different from the equations of tangent lines we have found before.

What would be a more practical example?

3) We can Linear Approximation to approximate square roots that would be difficult to find without the use of a calculator. Let's approximate $\sqrt{9.1}$. We will use the function $f(x) = \sqrt{x+8}$ at $a = 1$.

$f'(x) = \frac{1}{2}(x+8)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+8}}$. At $a = 1$, $f'(1) = \frac{1}{6}$ So $L(x) = 3 + \frac{1}{6}(x-1) \rightarrow$
$L(x) = \frac{1}{6}x + \frac{17}{6}$. So $\sqrt{9.1} \approx \frac{17}{6} + \frac{1}{6} \cdot 1.1 = 3.01\bar{6}$.

DIFFERENTIAL:



Notice $\Delta x = dx$ is the change in x, and we already know Δy is the change in y, or the change in height of the function over the interval $\Delta x = dx$. Now note, that dy is the change in height of the tangent line over the interval $\Delta x = dx$.

Recall the Linear Approximation (or equation of the tangent line) from the first part of this section:

$y - f(a) = f'(a)(x - a)$.

Let us also note that $x - a$ can be written as Δx (or change in x). Also note, the change in y is Δy, and can be written as $\Delta y = f(a + \Delta x) - f(a)$, at the value $x = a$, or more generally $\Delta y = f(x + \Delta x) - f(x)$.

We must understand that the differential is very small. In Physics, we use differentials often, and we often refer to them as differentially small. When Δx is differentially small, we define $\Delta x = dx$, and the differential becomes:

$\boldsymbol{dy = f'(x)dx}$.

Note that dy is the "dependent" variable, depending on both x and Δx (or dx).

We also observe that if we divide both sides by $dx \neq 0 \rightarrow \frac{dy}{dx} = f'(x)$. Whereas $f'(x)$ is the slope of the tangent line, dy is the change in the linearization when x changes by $\Delta x = dx$.

EXAMPLE:

1) Find the differential, dy, for $y = 3x^2$. $dy = 6xdx$.
   $(f'(x)dx)$

2) Find the differential for $y = \sqrt{e^x + x^3}$. $dy = \frac{1}{2}(e^x + x^3)^{-\frac{1}{2}}(e^x + 3x^2)dx = \frac{(e^x+3x^2)}{2\sqrt{e^x+x^3}}dx$.

3) Find the differential for $y = \sin^2 x - \tan^{-1} x$. $dy = \left(2 \sin x \cos x - \frac{1}{1+x^2}\right) dx$.

EXAMPLE:

Compare Δy with dy for $y = x^3 - 9x$, when x changes from 1 to 1.01:

First, $\Delta y = f(a + \Delta x) - f(a) \rightarrow \quad \Delta y = f(1.01) - f(1) = (1.01^3 - 9 \cdot 1.01) - (1^3 - 9 \cdot 1) = -0.059699$.

Second, $dy = f'(x)dx \rightarrow \quad dy = (3x^2 - 9)dx$. At $x = 1, dx = .01$, $dy = (3 - 9) \cdot (.01) = -.06$. So you can see that they are very close, but not identical, which is what we would expect.

EXAMPLE:

In this example, we will use the differential to perform error approximation. Let $A = \pi r^2$ be the area of a circular object we are trying to fit within a square frame. If the area has an error dA of approximately $.7\ cm^2$, what is the error in the radius? Will it fit in our $10 \times 10\ cm$ square frame, if the radius of our circle is $4.8\ cm$?

$dA = 2\pi r dr \rightarrow \quad .7 = 2\pi \cdot 4.8 \cdot dr \rightarrow \quad dr \approx .0232$. Our circle must have a radius no bigger than $5\ cm$, and this one will be no bigger than 4.823, so it will work.

HYPERBOLIC FUNCTIONS:

Hyperbolic functions are functions that are related to the hyperbola, similar to how trigonometric functions are related to the circle.

DEFINITION:

$\sinh x = \frac{e^{x}-e^{-x}}{2}$ (Note: We pronounce this as hyperbolic sine).

$$\frac{\cosh x = (e^{x}+e^{-x})}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

There are identities for these functions that are left as an exercise for the student.

Derivatives of Hyperbolic Functions:

Let us prove the first one: $d(\sinh x) = \cosh x$:

$\sinh x = \frac{e^{x}-e^{-x}}{2} \rightarrow \quad d(\sinh x) = \frac{(e^{x}+e^{-x})}{2}$. We immediately recognize this as $\cosh x$. The others can be proved similarly (and just as easily).

Derivatives of Hyperbolic Functions

(1) $(\sinh x)' = \cosh x$
(2) $(\cosh x)' = \sinh x$
(3) $(\tanh x)' = \sec h^{2} x$
(4) $(\coth x)' = -\csc h^{2} x$
(5) $(\sec hx)' = -\sec hx \tanh x$
(6) $(\csc hx)' = -\csc hx \coth x$

EXAMPLE:

Let $f(x) = \sinh 3x^{2}$. We use the Chain Rule: $f'(x) = (\cosh 3x^{2}) \cdot 6x = 6x \cdot \cosh 3x^{2}$.

EXERCISES:

Find the Linear approximation for the following functions at the given value for x:

1) $f(x) = e^x + 2x^2, \ x = 0$

2) $f(x) = \sin x + \tan^2 x, \ \ x = 0$

3) $f(x) = 3x^3 - x^{\frac{1}{3}}, \ \ x = 1$

4) Find the Linear approximation for $f(x) = \sqrt{x+1}$ for $a = 3$, and use it to approximate $\sqrt{4.01}$.

Find the differential:

5) $y = \sin^2 x \cos x$

6) $y = xe^x - \sqrt{x^2 - 1}$

7) $y = \ln(x^2) - \tan^{-1} x$

8) $y = \frac{x^4 - e^{2x}}{\cos^2 x - 2x}$

9) $y = \ln(\cos x \cdot \tan x)$

10) $y = \frac{20x^3 - \cosh x}{x^5 - \cos^3 x}$

Find the differential, dy, for the given values of x and dx:

11) $y = 3x^2 - 6x, \ \ x = 2, \ dx = 0.1$

12) $y = e^x - 9x, \ \ x = 0, \ dx = -0.2$

13) $y = \sin \pi x, \ \ x = \frac{1}{2}, \ \ dx = 0.01$

14) $y = \tan^2 \pi x, \ \ x = 1, \ \ dx = 0.03$

Compare Δy and dy for the following functions at the given values for x and $\Delta x = dx$: (For 15) and 16)):

15) $y = x^3 - 2,\ x = 3,\ \Delta x = 0.1$

16) $y = e^x + 1,\ x = 0,\ \Delta x = 0.2$

17) A circular disk has a radius of 3 cm, and a possible error of $0.02\ cm$. Use differentials to estimate the maximum error in its Area.

18) A sphere has a radius of 6 cm, and a possible error of $0.05\ cm$. Use differentials to estimate the maximum error in its Volume.

19) A cube with sides 5 cm, has a possible error of 0.1 cm. Use differentials to estimate the maximum error in its Surface Area.

CHAPTER 3
SECTION 1
MAXIMUM AND MINIMUM VALUES (EXTREMA)

We now venture into the territory of what derivatives can be used for. We have previously seen some examples with velocity and acceleration, as well as related rates. We will now use derivatives to find maxima and minima (generalized as extrema) of functions.

We will first discuss the absolute maximum, and absolute minimum of a function over a closed interval. Loosely speaking, the absolute maximum will be the highest point on the graph over the interval, and the absolute minimum will be the lowest point over the interval. For a continuous function, we will be guaranteed at least one of each (some functions may have one, or both extrema, that occur more than once).

DEFINITION:

Let f be a function defined on an interval I containing the number c.

1) $f(c)$ is the absolute minimum of f on I if $f(c) \leq f(x)$ for all x in I.

2) $f(c)$ is the absolute maximum of f on I if $f(c) \geq f(x)$ for all x in I.

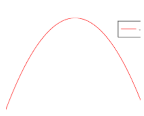
EXTREME VALUE THEOREM: This is the Theorem that guarantees at least one of each, over a closed interval for a continuous function (as mentioned above):

Let f be a continuous function on a closed interval $[a, b]$, then f is guaranteed an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ for some numbers c and d, that are both in $[a, b]$.

DEFINITION: Of local (or relative) extrema.

(Note: Some texts use local, and others use relative. They are interchangeable).

The number $f(c)$ is:

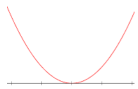
1) A local maximum of f if $f(c) \geq f(x)$ when x is near c. (i.e., it is a natural high spot, or like a hill). 

2) A local minimum of f if $f(c) \leq f(x)$ when x is near c. (i.e., it is a natural low spot, or like a valley). 

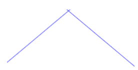
Note: The term "near c", means there exists an interval containing c for which it holds.

DEFINITION:

A **critical number** of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist. (This implies critical numbers are values at which the tangent line is horizontal, or there is no tangent line at that value).

Example 1: 

This one has a critical number at the bottom of the parabola. We can easily see that $f'(c) = 0$ here. (It has a horizontal tangent line).

Example 2: 

This one has a critical number at the top. We see here that $f'(c)$ does not exist. (We see we cannot get a tangent line at a sharp corner, which we have discussed previously).

FERMAT'S THEOREM: If f has a local maximum or minimum value at c, then c is a critical number of f, i.e. $f'(c) = 0$ or $f'(c)$ does not exist.

PROOF:

1) Let $f(c)$ be a local extremum and $f'(c)$ does not exist. Then by definition, it is a critical number.

2) Let $f(c)$ be a local extremum and $f'(c)$ does exist. Then $f'(c) < 0$, $f'(c) > 0$, or $f'(c) = 0$.

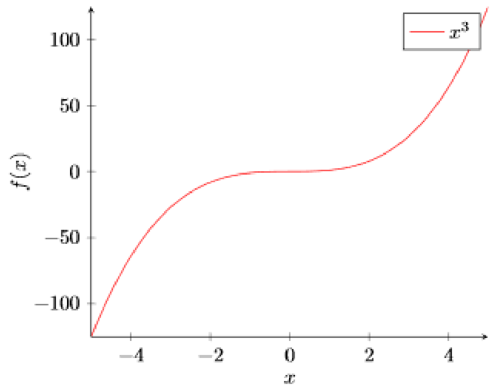
   Suppose $f'(c) > 0$, then there exists a c in (a, b) such that $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \neq c$ in (a, b). Because $\frac{f(x)-f(c)}{x-c} > 0$, both numerator and denominator must both be positive or must both be negative.

   Then, as $x \to c^- \to \quad x < c,\ and\ f(x) < f(c) \to\ f(c)$ is not a relative minimum.
   And, as $x \to c^+ \to \quad x > c,\ and\ f(x) > f(c) \to\ f(c)$ is not a relative maximum.

   The only other alternative that works is for $f'(c) = 0$.

   The case that $f'(c) < 0$ can be proved similarly.

   <u>Note</u>: **It is important to note that Fermat's Theorem is a "one-way" theorem. What does that mean? It means that it is if-then, not if-and-only-if. I.e., If f has a local extremum, then it is a critical number. It may not go the other way. I.e., if f has a critical number at c, it may not be a local extremum.**



Example:

Note in this example, $f(x) = x^3$ has a critical number at $x = 0$, where the slope of the tangent line is horizontal. Also note that it is simply a flat spot on the graph, and neither a local maximum or minimum.

EXAMPLE:

Find the critical numbers for the following functions:

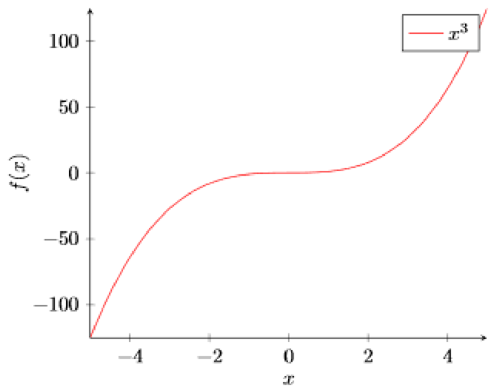
1) $f(x) = x^3 - 3x^2 - 9$: First we find $f'(x) = 3x^2 - 6x$. Next, we set $f'(x) = 0 \rightarrow$ $3x^2 - 6x = 0 \rightarrow \quad 3x(x-2) = 0 \rightarrow \quad x = 0$ or $x = 2$. These are our critical numbers.

2) $f(\theta) = \cos\theta + \theta$. $f'(\theta) = -\sin\theta + 1 = 0 \rightarrow \quad \sin\theta = 1 \rightarrow \quad \theta = \frac{\pi}{2} + 2n\pi$, where $n =$ $0, 1, 2, \ldots$

STEPS FOR FINDING THE ABSOLUTE EXTREMA ON A CLOSED INTERVAL:

1) Take the first derivative of f.
2) Find all critical numbers such that $f'(x) = 0$.
3) Find all critical numbers such that $f'(x)$ does not exist.
4) Substitute all critical numbers that are in (a,b), and the 2 endpoints of [a,b] into $f(x)$. **Note:** If you get values outside the interval [a,b], you must discard them as they do not apply.
   a) The largest value is the Absolute Maximum.
   b) The smallest value is the Absolute Minimum.
   Note: It is possible and viable to have more than one of each, i.e., the same value for f occurring at more than one value for x.

Example: Look again at the graph of $f(x) = x^3$. Note that over the closed interval $[-5,5]$, it has no critical values, but it has an absolute minimum at $x = -5$, and an absolute maximum at $x = 5$. We know that the Extreme Value Theorem guarantees at least one of each, and in this example, we clearly see they both occur at endpoints. We observe that endpoints, as well as critical numbers are candidates for absolute extrema.



EXAMPLE: Let $f(x) = x^2$ over the closed interval [-1,2].

1) $f'(x) = 2x$

2) $f'(x) = 0 \rightarrow \quad 2x = 0 \rightarrow \quad x = 0$

3) There no values for which $f'(x)$ does not exist.

4) $f(-1) = 1$

   $f(0) = 0$

   $f(2) = 4$

   We see that the absolute minimum is $f(0) = 0$, and the absolute maximum is $f(2) = 4$.

EXAMPLE: Let us find the Absolute Maximum and Absolute Minimum for the function we looked at previously: $f(x) = x^3 - 3x^2 - 9$. Let us confine it to the closed interval [-1,1].

1) We found $f'(x) = 3x^2 - 6x$

2) $f'(x) = 0 \rightarrow \quad 3x^2 - 6x = 0 \rightarrow \quad 3x(x-2) = 0 \rightarrow \quad x = 0$ or $x = 2$.

3) There no values for which $f'(x)$ does not exist.

4) $f(-1) = -13$

$f(0) = -9$

$f(2) = -13$

5) We note that the absolute minimum is both $f(-1) = -13$ and $f(2) = -13$, and the absolute maximum is $(0) = -9$.
We also observe that we discarded $x = 2$, as it is outside the interval [-1,1].

EXAMPLE: Let us find the Absolute Maximum and Absolute Minimum for the function we looked at previously: $f(\theta) = \cos\theta + \theta$. Let us confine it to the closed interval $[0, \pi]$.

1) We found $f'(\theta) = -\sin\theta + 1$

2) $f'(\theta) = -\sin\theta + 1 = 0 \rightarrow \ \sin\ \theta = 1 \rightarrow \ \ \theta = \frac{\pi}{2}$

3) There no values for which $f'(\theta)$ does not exist.

4) $f(0) = 1$

$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

$f(\pi) = -1 + \pi$

5) We note that $f(0) = 1$ is the absolute minimum, and $(\pi) = -1 + \pi$ is the absolute maximum.

EXAMPLE: Find the Absolute Maximum and the Absolute Minimum for $f(x) = \frac{1}{x^2+1}$ over the interval [-1,2]:

1) $f(x) = \frac{1}{x^2+1} = (x^2+1)^{-1} \rightarrow \quad f'(x) = -(x^2+1)^{-2} \cdot 2x = -\frac{2x}{(x^2+1)^2}$

2) $f'(x) = 0 \rightarrow \quad -\frac{2x}{(x^2+1)^2} = 0 \rightarrow \quad 2x = 0 \rightarrow \quad x = 0$

3) There no values for which $f'(x)$ does not exist. (Note: there are no real numbers for which $(x^2+1)^2 = 0$.

4) $f(-1) = \frac{1}{2}$

$f(0) = 1$

$f(2) = \dfrac{1}{5}$

5) We note that $f(2) = \frac{1}{5}$ is the absolute minimum, and $f(0) = 1$ is the absolute maximum.

EXAMPLE: Find the Absolute Maximum and the Absolute Minimum for $f(x) = x^{\frac{2}{3}}$ over the interval [-8,1]:

1) $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{\frac{1}{3}}}$

2) $f'(x) = 0 \rightarrow \frac{2}{3x^{\frac{1}{3}}} = 0 \rightarrow \quad 1 = 0$. So we see there are no solutions to this equation. This implies that there are no places on the graph such that the tangent line is horizontal.

3) This time we observe there is a number for which $f'(x)$ does not exist. We see that when $x = 0$, that $f'(x)$ is undefined, and therefore, does not exist. (We also observe that $f(0)$ does exist. If it did not, then $x = 0$ would not be a critical number as would not be in the domain of f).

4) $f(-8) = 4$

   $f(0) = 0$

   $f(1) = 1$

5) We observe that $f(-8) = 4$ is the absolute maximum, and $f(0) = 0$ is the absolute mimimum.

EXERCISES:

Find all critical numbers for the following functions:

1) $f(x) = 4x^2 - 12$
2) $f(x) = x^3 - 12x - 20$
3) $y = 3x^3 - 27x^2$
4) $y = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$
5) $y = e^x - x$
6) $f(x) = \cos x - x$
7) $f(t) = 14t - 2$
8) $f(x) = \frac{x+1}{x^2+1}$
9) $y = \tan^2 x$
10) $y = \ln(2x) - 2x^2$

Find the Absolute Maximum and the Absolute Minimum for the following functions over the given interval:

11) $f(x) = 2x^2 - 20,$ [-2,2]
12) $f(x) = 4x^2 - 2x + 7,\ [-1,1]$
13) $f(x) = 3x^2 - 5x - 17,\ [0,2]$
14) $f(x) = x^3 - 3x^2 - 2,\ [-1,4]$
15) $f(x) = -x^3 + 3x + 1,\ [-2,2]$
16) $f(x) = \frac{1}{3}x^3 - 2x + 5,\ [-1,3]$
17) $f(x) = x^3 - 6x^2 + 9x - 2,\ [-4,1]$
18) $f(x) = x^4 - 2x^2,\ [0,1]$

19) $f(x) = x^4 - 3x^3,\ [-1,4]$

20) $f(x) = 2\cos x,\ [0,2\pi]$:

21) $f(\theta) = 2\cos\theta - \theta,\ \ [0,\pi]$

22) $f(\theta) = 2\sin\theta + \sqrt{3}\theta,\ \ [0,2\pi]$

23) $f(x) = \frac{2}{x^2+2},\ [-2,3]$

24) $f(x) = \frac{1}{x^4+1},\ [-1,1]$

25) $f(x) = (x+1)^{\frac{2}{3}},\ [-2,2]$

26) $f(x) = (x-2)^{\frac{2}{3}},\ [0,3]$

27) $f(x) = -3(x-1)^{\frac{2}{3}} + 2,\ [0,2]$

28) $f(t) = e^t - t,\ [-1,1]$

CHAPTER 3
SECTION 2
THE MEAN VALUE THEOREM

The idea behind the Mean Value Theorem is super cool! Recall from Chapter 1, Section 6, we introduced the idea of instantaneous rates of change, and we talked about a skier skiing down a mountain. We introduced the derivative, at this time, as an instantaneous rate of change (the slope of the tangent line). We discussed the difference between an average speed $\left(\frac{Total\ Distance}{Total\ Time}\right)$, vs an instantaneous speed at a particular point in time. If you are a skier, do you think your average speed will always be the same as your instantaneous speed? Most likely not. As the terrain changes, you will speed up and slow down accordingly. You may have to stop to avoid someone. You may speed up to get out of the way to avoid someone else. It would be very unlikely that you would always be skiing at a constant speed. What the Mean Value Theorem gives us, is that if you can model your distance vs time as a continuous and differentiable function, then at least once, your instantaneous speed will equal your average speed. (This applies to other rates of change as well).

We will prove the Mean Value Theorem, but before we can do that, we must prove a smaller theorem called Rolle's Theorem.

ROLLE'S THEOREM:

Let f be a function that satisfies the following:

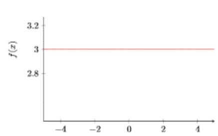
1) f is continuous on the closed interval $[a, b]$.
2) f is differentialbe on the open interval (a, b).
3) $f(a) = f(b)$

Then there exists a number c in (a, b) such that $f'(c) = 0$.

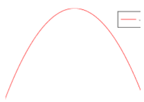
PROOF:

We will prove this with 3 cases:

1) $f(x) = k$, a constant. In this case $f'(x) = 0$ everywhere, so c is any number in (a, b).



2) $f(x) > f(a)$ for some $x\ in\ (a, b)$. By the Extreme Value Theorem, $f(x)$ has an absolute maximum in $[a, b]$. Since $f(a) = f(b)$, it must occur within (a, b), and not at the endpoints. Therefore, f has a local maximum at c, and $f'(c) = 0$ by Fermat's Theorem.



3) Similarly to the above argument, if $f(x) < f(a)$, for some value $x \ in \ (a,b)$, f has a local minimum in (a,b), and $f'(c) = 0$ by Fermat's Theorem, using the same argument in 2).

EXAMPLE:

Let $f(x) = x^2 + 1$, $[-2,2]$. We will use Rolle's Theorem to find all numbers c that satisfy the conclusion of Rolle's Theorem. We start by showing this function satisfies the 3 conditions of Rolle's Theorem:

1) f is continuous on $[-2,2]$. True, polynomials are continuous everywhere.
2) f is differentiable on $(-2,2)$. True, polynomials are differentiable everywhere.
3) $f(-2) = 5$, $f(2) = 5$. This condition holds true.

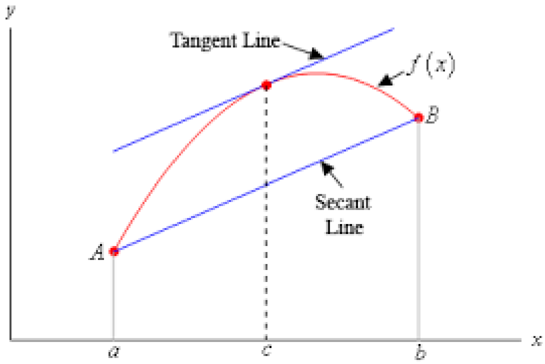
Then, there exists a c such that $f'(c) = 0$ in (a,b). $f'(x) = 2x = 0 \rightarrow \quad x = 0$. 0 is in the open interval $(-2,2)$. This is only value for c that we obtain.

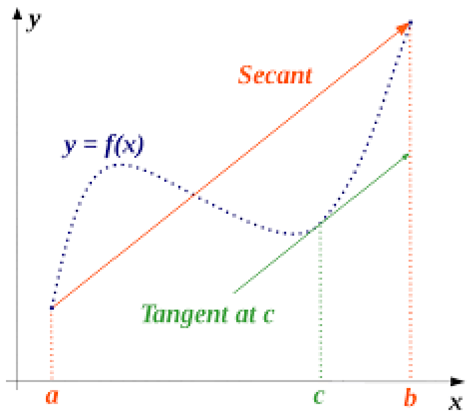
<u>THE MEAN VALUE THEOREM:</u>

Let f be a function that satisfies the following conditions:

1) f is continuous on the closed interval $[a,b]$.
2) f is differentialbe on the open interval (a,b).

Then there exists a number c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Note that $f'(c)$ is our derivative, our instantaneous rate of change, and that $\frac{f(b)-f(a)}{b-a}$ is our average rate of change. We observe that it is exactly what we talked about in our ski example.





PROOF:

To prove the Mean Value Theorem, we will have to use Rolle's Theorem in our proof.

Let us first write our the equation of a secant line at a point from $x = a$ to $x = b$.

$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$. Or solving for y: $y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$.

The next thing we will do is create a new function $h(x)$. THIS IS A TRICK: We define $h(x)$ to be the difference between $f(x)$ and the secant line function we expressed above:

$$h(x) = f(x) - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right) \rightarrow \; h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Next we will use Rolle's Theorem: To do that, we must first verify it meets all 3 conditions:

1) $h(x)$ is continuous on $[a, b]$, since f is continuous (as stated in the first condition for the Mean Value Theorem, and $y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$, is the equation of a line, which is continuous everywhere.

2) $h(x)$ is differentiable on (a, b), since f is differentiable (as stated in the second condition for the Mean Value Theorem, and $y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$, is the equation of a line (not vertical), which is differentiable everywhere.

3) We must now show that $h(a) = h(b)$. $h(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0 - 0 = 0$.

$$h(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = f(b) - f(a) - \big(f(b) - f(a)\big) = 0.$$

We can clearly see that Rolle's Theorem is satisfied: Therefore, there exists a number c in (a, b) such that $h'(c) = 0$.

$h'(x) = f'(x) - 0 - \frac{f(b)-f(a)}{b-a} \cdot 1 = 0 \rightarrow \quad h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}$.

(Note that the derivative of $f(a)$ is 0, because it is a constant, and the derivative of $x - a$ is 1, because a is a constant).

EXAMPLE:

Let $f(x) = x^3 - 12x$, $[-1,2]$.

$f(x)$ is continuous on $[-1,2]$, and differentiable on $(-1,2)$, since it is a polynomial and both continuous and differentiable everywhere.

Therefore, the Mean Value Theorem applies and there exists a number c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Let us find all values c, for which this holds.

$f'(x) = 3x^2 - 12$. $\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(-1)}{2-(-1)} = \frac{-16-11}{2+1} = \frac{27}{3} = 9$. $f'(c) = 3c^2 - 12 = 9 \rightarrow \quad c^2 = \frac{21}{3} = 7$.

Therefore $c = \pm\sqrt{7}$. We found two values that work!

EXAMPLE:

You are skiing down a mountain with an average speed of 30 mph. Your distance vs time is modeled by the function $f(t) = \frac{1}{3}t^3 - \frac{7}{2}t$. You skied for 5 minutes. How many minutes after you began your descent , did you go 30 mph?

First, we must convert 30 mph to miles/minute. We get 1/2 miles/minute, or .5 miles/minute. We observe that $f(t)$ is a polynomial, and therefore, continuous on $[0,5]$, and differentiable on $(0,5)$.

$f'(t) = t^2 - \frac{7}{2}$. So $f'(c) = c^2 - \frac{7}{2}$. We also know that . $\frac{f(b)-f(a)}{b-a} = \frac{1}{2}$ in miles per minute. So

$c^2 - \frac{7}{2} = \frac{1}{2} \rightarrow \quad c^2 = 4 \rightarrow \quad c = \pm 2$. Of course time cannot be negative, so the only time during our run that we skied 30 mph was at $t = 2$ minutes after the beginning of our descent.

We get some other results from the Mean Value Theorem as well. One is the following theorem:

THEOREM:

If $f'(x) = 0$ for all x in an open interval $I = (a,b)$, then $f(x)$ is constant on I.

This makes intuitive sense, since $f'(x)$ is the slope of the tangent line. If $f'(x) = 0$, the slope of the tangent line is horizontal on all of I. Therefore, $f(x)$ would be constant. But, we can also use the Mean Value Theorem to prove it:

PROOF:

Let x_1, x_2 be any two numbers in I. Also let $x_1 < x_2$. Since we are using the Mean Value Theorem, f is continuous on $[a, b]$, and differentiable on (a, b). Therefore, f will be continuous on $[x_1, x_2]$, and differentiable on (x_1, x_2), since x_1, x_2 are both in $I = (a, b)$. Therefore, the Mean Value Theorem applies: Therefore, we have a number c such that $x_1 < c < x_2$ such that:

$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \rightarrow \quad f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since, $f'(x) = 0$, for all x in I, then $f'(c) = 0$, and then $f(x_2) - f(x_1) = 0 \rightarrow \quad f(x_2) = f(x_1)$. Therefore, f is constant on I.

COROLLARY:

If $f(x) = g(x)$ for all x in I, then $f - g$ is constant on I. This implies, $f(x) = g(x) + c$, where c is a constant.

EXERCISES:

Sketch the graph of the following functions over the given interval. Sketch the slope of the secant line over the interval, then sketch the tangent lines with the same slope as the secant line: (You can use a graphing calculator if needed).

1) $f(x) = 2x^2 - 4,\ [-1,3]$

2) $f(x) = x^3 - 2x^2 + 9,$ [-3,4]

3) $f(x) = e^x, [0,5]$

4) $f(x) = \sin x,\ \left[0, \frac{3\pi}{2}\right]$

Verify that the function satisfies the 3 conditions for Rolle's Theorem over the given interval. Then find all values c that satisfy its conclusion.

5) $f(x) = x^2 - 9,\ [-2,2]$

6) $f(x) = x^3 + x^2,\ [-1,0]$

7) $f(x) = \sin 2x,\ [0, \pi]$

8) $f(x) = e^{x^2},\ [-1,1]$

Verify that the function satisfies the 2 conditions for the Mean Value Theorem over the given interval. Then find all values c that satisfy its conclusion:

9) $f(x) = 3x^2 + 2x - 3,\ [0,1]$

10) $f(x) = x^3 - 3x^2 + 4,\ [-1,1]$

11) $f(x) = 3\ln 2x,\ [1,2]$

12) $f(x) = \cos x,\ [0, \pi]$

13) You are skiing down a mountain with an average speed of 20 mph. Your distance vs time is modeled by the function $f(t) = \frac{1}{3}t^3 - \frac{7}{3}t$. You skied for 8 minutes. . How many minutes after you began your descent, did you go 20 mph?

14) You are on a road trip. You drove for 5 hours today. Your average speed was 60 mph. Your distance vs time is modeled by the function $f(t) = t^4 - 8t$. How many minutes after you began your road trip, did you go 60 mph?

15) You are traveling in a car. You travel for 90 miles in 2 hours. Prove that your speed was 45 mph at least once. (Your model function is continuous and differentiable everywhere).

16) You are running a foot race. At $t = 10$ minutes into the race, your speed is 10 mph. At $t = 30$ minutes in, your speed is 15 mph. Show that at some time during the race, your acceleration is $\frac{1}{4}\frac{miles}{hour^2}$.

CHAPTER 3
SECTION 3
HOW DERIVATIVES AFFECT THE SHAPE OF A GRAPH

Derivatives affect the shape of a graph in many ways. "First Derivatives" show you where the function is increasing, where it is decreasing, and where it has a local maximum and/or local minimum (which we discovered in Section 1 of this chapter). "Second Derivatives" show you where the function is concave up or down (which we will define in this section, but it basically refers to where a curve opens up or down); along with inflection points, where the function changes concavity.

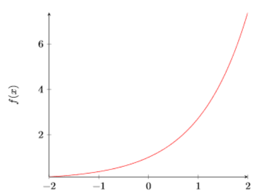
This section leads us into how to sketch curves using derivatives (a great application of derivatives, as graphs model real-world ideas), and will further lead us into applications referred to as optimization.

At this point in our journey, we should be getting excited about what lies ahead, and how Calculus shapes our daily lives and the world around us.

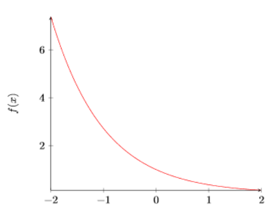
We will begin this section by defining Increasing and Decreasing functions.

DEFINITION:

1) A function f is said to be increasing over an interval I if for every a and b in I, and if $a < b$ then $f(a) < f(b)$. (In Laymen's terms this means, as we move from left to right, f gets bigger: goes up).



2) A function f is said to be decreasing over an interval I if for every a and b in I, and if $a < b$ then $f(a) > f(b)$. (In Laymen's terms this means, as we move from left to right, f gets smaller: goes down).



INCREASING/DECREASING TEST/ THEOREM:

Let f be a continuous function on [a,b], and differentiable on (a,b). Then

1) If $\boldsymbol{f'(x) > 0}$ for all x in (a,b), then f is **increasing** on [a,b]. Intuitively, this makes sense. We know that the derivative represents the slope of the tangent line, and if the slope of the tangent line is positive, the function is increasing.

2) If $\boldsymbol{f'(x) < 0}$ for all x in (a,b), then f is **decreasing** on [a,b]. Intuitively, this also makes sense. Again, since the derivative represents the slope of the tangent line, and if the slope of the tangent line is negative, it is decreasing.

PROOF: (OF NUMBER 1)

Let $f'(x) > 0$ for all x in (a,b), and $x_1 < x_2$ be two values in (a,b). By the Mean Value Theorem, there exists a number c in (x_1, x_2) such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Since $x_1 < x_2$, then $x_2 - x_1$ is positive, and $f'(c)$ is also positive (given in our statement), then $f(x_2) - f(x_1)$ must also be positive. Therefore, f is increasing on [a,b].

Number 2) can be proved similarly.

LOCAL EXTREMA:

We discussed local extrema in Section 1 of this chapter. These are local (or relative) maximum, and local (or relative) minimum values. Recall that they only occurred at critical points (where $f'(x) = 0$ or where $f'(x)$ does not exist (by Fermat's Theorem)). We also recall that not every critical point is a local extrema (recall the function $f(x) = x^3$ which has a critical number at $x = 0.$) Note: I used the words critical points and critical numbers here. A point is an ordered pair (x, y). A critical number is one value, in our case $x = c$, where c is a constant.

So once we find our critical points, how will we determine whether it is a local maximum, a local minimum or neither? We will have two ways. The first way is to use the first derivative.

THE FIRST DERIVATIVE TEST:

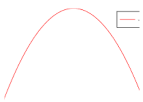
Let c be a critical number of a continuous function f

1) If f is increasing before c, and decreasing after c, (i.e., positive before and negative after), f has a local maximum at $(c, f(c))$.

2) If f is decreasing before c, and increasing after c, (i.e., negative before and positive after), f has a local minimum at $(c, f(c))$.

3) If f does not change signs at c, then $(c, f(c))$ is not a local maximum or local minimum. (It will merely be a flat spot on the graph like $c = 0$ for $f(x) = x^3$).

EXAMPLE:

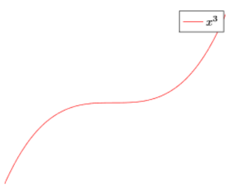


Notice this graph has a critical point $(c, f(c))$ at its center where $f'(x) = 0$. Also notice it is increasing before c and decreasing after c. Therefore, $(c, f(c))$ is a local maximum.

EXAMPLE:

Notice this graph has a critical point $(c, f(c))$ at its center where $f'(x) = 0$. Also notice it is decreasing before c and increasing after c. Therefore, $(c, f(c))$ is a local minimum.

EXAMPLE:



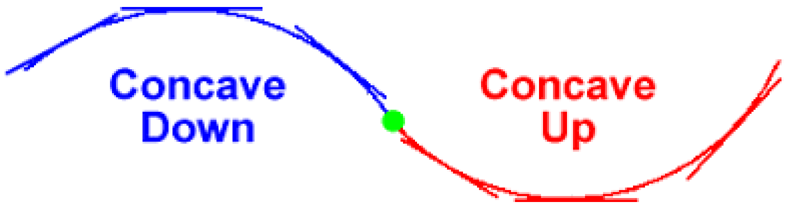
Notice this graph has a critical point $(c, f(c))$ at its center where $f'(x) = 0$. Also notice it is increasing before c and increasing after c. Therefore, $(c, f(c))$ is a not a local extrema. (We see it is a flat spot on the graph).

EXAMPLE;

Let $f(x) = x^2 - 2$. Find all local maximum and minimum values. $f'(x) = 0 \rightarrow \quad 2x = 0 \rightarrow \quad x = 0.$ Since $x = 0$ is the only critical number of f, we can substitute any value on either side to see where it is increasing and decreasing. $f'(-1) = -2,\ f'(1) = 2$. So we see it is decreasing before $x = 0$, and increasing after. Therefore, $(0, -2)$ is a local minimum value. (Note: How did we find the -2? We always use $f(c)$ to find the y-coordinate).

<u>HOW SECOND DERIVATIVES AFFECT THE SHAPE OF A GRAPH:</u>

Take a look at the graph below:



Notice the blue graph labeled Concave Down. Also notice the tangent lines drawn on the graph. Since the first derivative shows us where the function is increasing and decreasing, the second derivative shows us where the derivative is increasing or decreasing. Notice on the blue graph, the slopes of the tangent lines are always decreasing. They start out positive, go to zero, and become negative. Therefore $f''(x) < 0$, since the slopes of the tangent lines are always decreasing. (Note: Do not confuse this for where the function itself is increasing or decreasing. This time we are observing what the derivative itself is doing).

Similarly, we look at the red graph labeled Concave Up. Notice on this graph, the slopes of the tangent lines are always increasing. They start out negative, go to zero, and become positive. Therefore $f''(x) > 0$, since the slopes of the tangent lines are always increasing.

To remember which is which, we say **Concave Up Holds Water, and Concave Down Spills Water.**

CONCAVITY TEST:

1) If $f''(x) > 0$ for all x in an interval I, then $f(x)$ is concave up over I.

2) If $f''(x) < 0$ for all x in an interval I, then $f(x)$ is concave down over I.

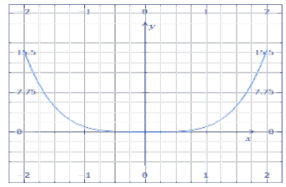
DEFINITION:

A point $(c, (f(c))$ is called an **Inflection point**, if f is continuous at c, and f changes concavity at $(c, (f(c))$ (meaning from concave up to concave down, or vice versa).

THEOREM: TO FIND AN INFLECTION POINT:

An inflection point $(c, f(c))$ will only occur if $f''(c) = 0$ or if $f''(c)$ does not exist.

(Note that this is a one way theorem. If there is an inflection point, it must satisfy the above conditions; but conversely just because $f''(c) = 0$ or if $f''(c)$ does not exist, does not necessarily mean there is an inflection point there. Example: $f(x) = x^4$. $f''(x) = 12x^2$. $12x^2 = 0 \rightarrow \quad x = 0$. But $x = 0$ is not an

inflection point, as $f(x) = x^4$ is concave up everywhere. 

EXAMPLE:

Let $f(x) = x^3 - 6x^2 + 2$. Let's find all inflection points, and the intervals of concavity for this function.

First, we must find $f'(x)$ and $f''(x)$. $f'(x) = 3x^2 - 12x \rightarrow \quad f''(x) = 6x - 12$.

Next we find the inflection point(s) by setting $f''(x) = 0$. $6x - 12 = 0 \rightarrow \quad x = 2$, gives $(2, -14)$ as our only possible inflection point. (Note: again "point" infers both an x and y coordinate). This is our only inflection point. (Note there are no cases where $f''(x)$ does not exist).

So now we need to only choose any number on either side of $x = 2$, and substitute each into $f''(x)$. We choose $x = 0$, and $x = 3$: $f''(0) = -12$, $f''(3) = 6$.

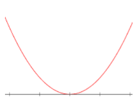
Therefore $f(x)$ is concave down from $(-\infty, 2)$, and concave up from $(2, \infty)$, which gives us $(2, -14)$ as our only inflection point.

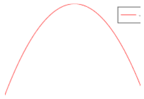
THE SECOND DERIVATIVE TEST:

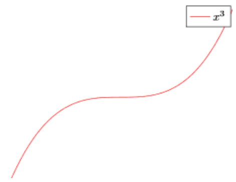
Recall, previously we mentioned there would be two ways to find out if a critical point was a local maximum, a local minimum, or neither. The first method was using the First Derivative Test to see where the function was increasing and decreasing. The next method will be to use the Second Derivative Test to see if a function is a local maximum or a local minimum. Sometimes this test fails. In this instance, it is necessary to use the First Derivative Test.

It goes as follows:

Let f'' be continuous near $x = c$:

1) If $f'(c) = 0$, and $f''(c) > 0$, f has a local minimum at $(c, f(c))$. (Note that f is concave up here). 

2) If $f'(c) = 0$, and $f''(c) < 0$, f has a local maximum at $(c, f(c))$. (Note that f is concave down here). 

3) If $f'(c) = 0$, and $f''(c) = 0$, or if $f''(c)$ does not exist, the test fails. f could be a local minimum, a local maximum, or neither. E.g., it could look like this: 

EXAMPLE: Let's finish our previous example: $f(x) = x^3 - 6x^2 + 2$. Let us find:

a) The intervals for which f is increasing or decreasing
b) The local maximum and minimum values of f
c) The inflection point(s) and intervals of concavity.

(Note that we have already found part c).

a) First we find all critical points: $f'(x) = 0 \rightarrow \quad 3x^2 - 12x = 0 \rightarrow \quad x = 0{,}4$. So we get $(0{,}2)$ and $(4, -30)$ as our critical points. We need to substitute values less than 0, between 0 and 4, and greater than 4 into the first derivative. $f'(-1) = 15$, $f'(1) = -9$, $f'(5) = 15$. Therefore f is increasing from $(-\infty, 0) \cup (4, \infty)$, and decreasing from $(0{,}4)$.

b) It is easy to see that $(0{,}2)$ is a local maximum since it was increasing before, and decreasing after. We also observe that from our work in the previous example, that it was concave down there.

   We also observe that $(4, -30)$ is a local minimum since it was decreasing before, and increasing after. We also observe that from our work in the previous example, that it was concave up there.

c) We already found this information in our previous example.

EXAMPLE:

Let $f(x) = 2\sin 2x \text{ from } [0, \pi]$. Again, let us find:

a) The intervals for which f is increasing or decreasing
b) The local maximum and minimum values of f
c) The inflection point(s) and intervals of concavity

a) First we find all critical points: $f'(x) = 0 \rightarrow \;\; 4\cos 2x = 0 \rightarrow \quad x = \frac{\pi}{4}$. So our only critical point is $\left(\frac{\pi}{4}, 2\right)$. So we need only substitute vales on either side of $\frac{\pi}{4}$ and inside $(0, \pi)$ into $f'(x)$ to find where f is increasing or decreasing. $f'\left(\frac{\pi}{6}\right) = 2, \;\; f'\left(\frac{\pi}{3}\right) = -2$. Therefore, f is increasing from $\left(-\infty, \frac{\pi}{4}\right)$ and decreasing from $\left(\frac{\pi}{4}, \infty\right)$.

b) Since f is increasing before $x = \frac{\pi}{4}$, and decreasing after, $\left(\frac{\pi}{4}, 2\right)$ is a local maximum.

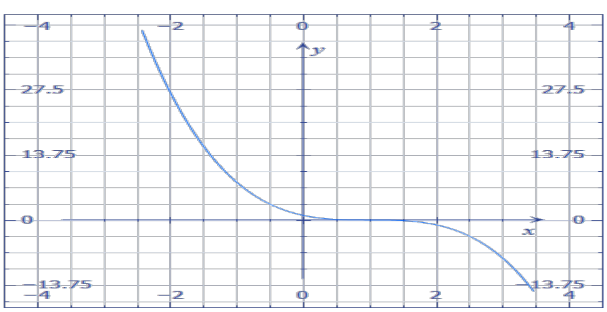
c) To find all inflection point(s), we set $f''(x) = 0$. (Note there are no cases where $f''(x)$ does not exist).

$f''(x) = -8\sin 2x. \;\; -8\sin 2x = 0 \rightarrow \quad x = 0, \frac{\pi}{2}$. Since 0 is an endpoint, it is not considered an inflection point on our interval, since it is not in our selected domain. Therefore, we only consider $x = \frac{\pi}{2}$. Next, we need to substitute values on either side of $\frac{\pi}{2}$ and inside $(0, \pi)$ into $f''(x)$ to see where f is concave up or concave down.
$f''\left(\frac{\pi}{4}\right) = -8, \;\; f''\left(\frac{3\pi}{4}\right) = 8$. Therefore, we conclude that f is concave down from $\left(-\infty, \frac{\pi}{2}\right)$, and concave up from $\left(\frac{\pi}{2}, \infty\right)$.

EXAMPLE:

Let us now construct a graph given certain conditions of derivatives. (Note that a graph that meets the conditions given is not necessarily unique). There may be an infinite number of possibilities. We only need to construct a graph that meets the given conditions. Any number of graphs may be different.

Let us sketch a graph that has f is decreasing and concave up on $(-\infty, 1)$ and decreasing and concave down on $(1, \infty)$.



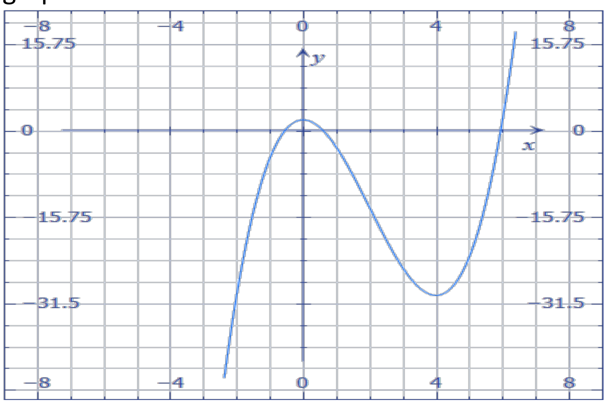
Note how this graph satisfies all conditions given. Also notice it is not unique.

EXAMPLE:

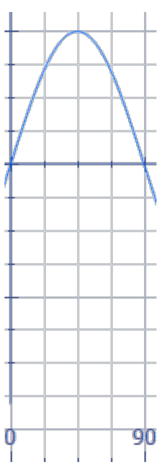
For the functions given, find the following:

a) The intervals for which f is increasing or decreasing
b) The local maximum and minimum values of f
c) The inflection point(s) and intervals of concavity
d) Use steps a)-c) to sketch the graph of the function:

1) $f(x) = x^3 - 6x^2 + 2$. We have already completed steps a)-c) above. We will now sketch the graph:



2) Let $f(x) = 2\sin 2x$ from $[0, \pi]$. Again, we already found parts a)-c) above. We will now sketch the graph.



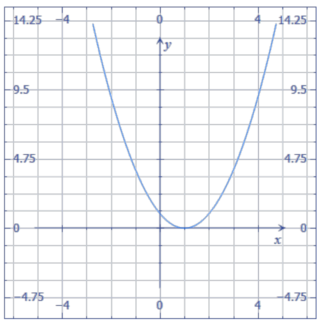
3) Let $f(x) = x^2 - 2x + 1$.

a) Let us first find all critical points of f: $f'(x) = 2x - 2 = 0 \rightarrow \quad x = 1$. So $(1,0)$ is our only critical point. $f'(0) = -2, \;\; f'(2) = 2$. Therefore, f is decreasing from $(-\infty, 1)$ and increasing from $(1, \infty)$.

b) Since f is decreasing before $x = 1$, and increasing after, f has a local minimum at (1,0).

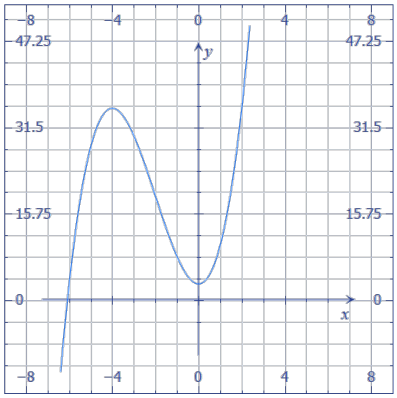
c) To find inflection point(s): $f''(x) = 0 \rightarrow \quad 2 = 0$. This is an untrue statement. Therefore, there are no infection points. $f''(x) = 2$ is constant and positive. Therefore f is concave up $(-\infty, \infty)$.

d) To sketch:



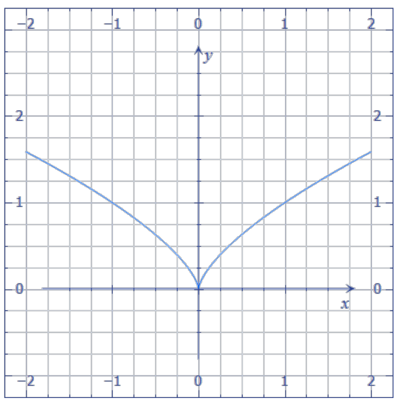
EXERCISES:

1) Use the graph below to find the following:



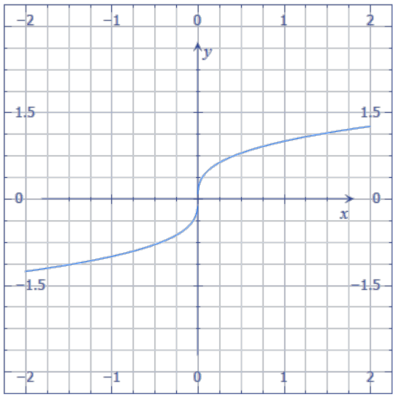
a) The location of any local maximum and local minimum values: Give x-values, and you can approximate the y-values.
b) The intervals for which f is increasing, and where f is decreasing.
c) Approximate any inflection points.
d) Find all intervals of concavity.

2) Use the graph below to find the following:



a) The location of any local maximum and local minimum values:
b) The intervals for which f is increasing, and where f is decreasing.
c) Find any inflection points.
d) Find all intervals of concavity.

3) Use the graph below to find the following:



a) The location of any local maximum and local minimum values:
b) The intervals for which f is increasing, and where f is decreasing.
c) Find any inflection points.
d) Find all intervals of concavity.

Use the following information to sketch a graph that meets the conditions given. Note that your graph can be correct without being unique:

4) f is decreasing and concave up on $(-\infty, 3)$ and decreasing and concave down on $(3, \infty)$.

5) f is increasing and concave up on $(-\infty, 1)$ and increasing and concave down on $(1, \infty)$.

6) f is concave up from $(-\infty, 2)$ and concave down from $(2, \infty)$ and has an inflection point at $(2,3)$.

7) f has a local maximum at $(2,5)$, is concave down from $(-\infty, 4)$, has a vertical asymptote at $x = 4$, and concave up from $(4, \infty)$.

8) $f(1) = 1$, f is concave up everywhere, and f has a local minimum at $(0, -1)$.

9) $f(0) = 0$, $f(1) = -2$, $f(-1) = 2$. f has a local minimum at $(1, -2)$, a local maximum at $(-1,2)$, an infection point at $(0,0)$.

For the functions given, find the following:

a) The intervals for which f is increasing or decreasing
b) The local maximum and minimum values of f
c) The inflection point(s) and intervals of concavity
d) Use steps a)-c) to sketch the graph of the function:

10) $f(x) = 2x^2 - 8x + 1$:

11) $f(x) = x^2 - 6x + 2$:

12) $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$:

13) $f(x) = x^3 - 27x$

14) $f(x) = -x^3 + 3x^2 + 2$

15) $f(x) = x^3 + 6x^2 - 1$

16) $f(x) = x^4 - 4x^3 + 4x^2$

17) $f(x) = x^4 - 4x^3$

18) $f(x) = 2\cos 3x$

19) $f(x) = \sin x + \cos x;\ \ [0,2\pi]$

20) $f(x) = x^{\frac{1}{3}}$

21) $f(x) = x^{\frac{2}{3}}$

CHAPTER 3
SECTION 4
SUMMARY OF CURVE SKETCHING

In the previous section, we learned how first and second derivatives affect the shape of a graph. We learned how first derivatives show us where a function is increasing, and where it is decreasing; and how to find critical points and local extrema. We also learned how second derivatives help us find inflection points, and where a function is concave up and concave down. In Chapter 1, Section 1, we learned about infinite limits and vertical asymptotes; and in Chapter 1, Section 5, we learned about limits at infinity and horizontal asymptotes. In this section, we will put it all together to sketch a variety of graphs.

Steps for Curve Sketching:

a) x and y intercepts
b) Vertical and horizontal asymptotes
c) 1st and 2nd derivatives
d) Critical points
e) Intervals of Increase or decrease
f) Local/Relative extrema
g) Inflection points
h) Intervals of concavity
i) Sketch

a) x and y intercepts: Recall what x and y intercepts are: They are where the function crosses the x-axis, and where it crosses the y-axis. To find each, set the other variable equal to 0, and solve.

b) Asymptotes: Recall, from Section 1.1: To find the vertical asymptote(s): Set the denominator equal to zero, and solver for x. (If it doesn't cancel a factor in the numerator, it's a vertical asymptote). From Section 1.5: To find the horizontal asymptote: Find $\lim_{x \to \pm\infty} f(x)$. (Simplify, if necessary, and the limit will be the horizontal asymptote). We will also briefly discuss slant asymptotes at the end of this section.

c) 1st and 2nd derivatives: Find $f'(x)$ and $f''(x)$ using all the rules we have learned.

d) Critical points: Set $f'(x) = 0$, solve for x, and find the y-value as well. Also check to see if there is a value for which $f'(x)$ does not exist, yet $f(x)$ does exist. If there is a critical number, also find the y-value.

e) Intervals of Increase or decrease: Substitute a value between each critical point into $f'(x)$. If it is positive, it is a region that is increasing; and if negative, decreasing.

f) Local/Relative extrema: Use part e) to check on either side of each critical point. If it is increasing before and decreasing after, it is a local maximum. If it is decreasing before and increasing after, it is a local minimum. You can also use the second derivative. You will substitute a critical number into the second derivative. If it is concave down, it is a maximum. If it is concave up, it is a minimum. Recall that if $f''(x) = 0$, the second derivative test fails.

g) Inflection points: Solve for x by setting $f''(x) = 0$, and where $f''(x)$ does not exist; and check to see if the concavity changes on either side.

h) Intervals of concavity: Generalize your findings from the previous step, and put it in interval notation for where f is concave upward, and where it is concave downward.

i) Sketch: Use all the steps above to create the sketch of your graph.

EXAMPLE:

Let $f(x) = x^3 - 3x + 2$. Let us sketch the curve using the above steps:

a) x and y intercepts: To find the x-intercept(s): $x^3 - 3x + 2 = 0 \rightarrow \quad x = 1, -2$. To find the y-intercept: $0^3 - 3 \cdot 0 + 2 = 2 \rightarrow \quad y = 2$.

b) Vertical and horizontal asymptotes: f is a polynomial, and has no vertical asymptotes (it is continuous everywhere). It also has no horizontal asymptotes (it has no denominator, or more precisely, the denominator is 1).

c) 1st and 2nd derivatives: $f'(x) = 3x^2 - 3, \;\; f''(x) = 6x$

d) Critical points: We set $f'(x) = 0 \rightarrow \quad 3x^2 - 3 = 0 \rightarrow \quad x = \pm 1$, gives $(-1,4), (1,0)$. Since f is a polynomial, it is differentiable everywhere, so there is no value such that $f'(x)$ does not exist.

e) Intervals of increase or decrease: We choose 3 values: Something less than -1, something in the interval $(-1,1)$, and something greater than 1: $f'(-2) = 9$, so f is increasing from $(-\infty, -1)$. $f'(0) = -3$, so f is decreasing from $(-1,1)$. $f'(2) = 9$, so f is increasing from $(1, \infty)$. We summarize :

f is increasing from $(-\infty, -1) \cup (1, \infty)$.
f is decreasing from $(-1,1)$.

f) Local Extrema: Our only critical points are: $(-1,4), (1,0)$. f was increasing before $x = -1$, and decreasing after. Therefore, $(-1,4)$ is a local maximum. f was decreasing before $x = 1$, and increasing after. Therefore $(1,0)$ is a local minimum.

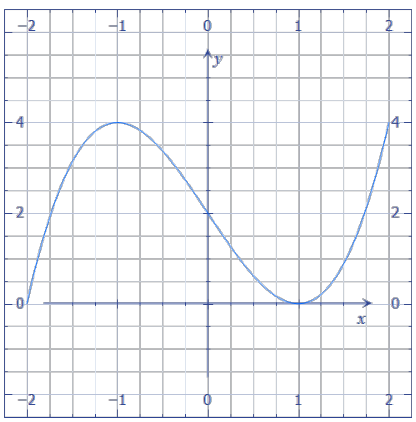
g) Inflection points: $f''(x) = 0 \rightarrow \quad 6x = 0 \rightarrow \quad x = 0, \ (0,2)$. We check to see if f changes concavity there: $f''(-1) = -6, \ f''(1) = 6$. f changes concavity at $(0,2)$. Therefore $(0,2)$ is our only inflection point. (Note: There are no values for which $f''(x)$ does not exist).

h) Intervals of concavity: Now, we generalize what we found in part g):

f is concave down from $(-\infty, 0)$ (since $f''(-1) = -6$)
f is concave up from $(0, \infty)$ (since $f''(1) = 6$)

(We also note here that since $f''(-1)$ is concave down, it shows $(-1,4)$ is a local maximum, and since $f''(1)$ is concave up, it shows $(1,0)$ is a local minimum. It is another way to find part f)).

i) Sketch:



EXAMPLE:

Let $f(x) = \sin x + \cos x, [0,2\pi]$. Let us again sketch the curve by finding the steps above:

a) x and y intercepts: x-intercept(s): $\sin x + \cos x = 0 \rightarrow \quad \sin x = -\cos x \rightarrow \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$. To find the y-intercept: $\sin 0 + \cos 0 = 1$. So $y = 1$.

b) Vertical and horizontal asymptotes: There are no vertical or horizontal asymptotes:

c) 1^{st} and 2^{nd} derivatives: $f'(x) = \cos x - \sin x, \ \ f''(x) = -\sin x - \cos x$.

d) Critical points: We set $f'(x) = 0 \rightarrow \quad \cos x - \sin x = 0 \rightarrow \quad \cos x = \sin x \rightarrow \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$ gives $\left(\frac{\pi}{4}, \sqrt{2}\right), \left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

e) Intervals of increase or decrease: We choose 3 values: Something less than $\frac{\pi}{4}$, something in the interval $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$, and something greater than $\frac{5\pi}{4}$: $f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}$ is positive. Therefore f is increasing from $\left(0, \frac{\pi}{4}\right)$. $f'\left(\frac{\pi}{2}\right) = -1$, so f is decreasing from $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$. $f'\left(\frac{3\pi}{2}\right) = 1$, so f is increasing from $\left(\frac{5\pi}{4}, 2\pi\right)$. We summarize:

f is increasing from $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$
f is decreasing from $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

f) Local Extrema: Our only critical points are: $\left(\frac{\pi}{4}, \sqrt{2}\right), \left(\frac{5\pi}{4}, -\sqrt{2}\right)$. f is increasing before $\frac{\pi}{4}$ and decreasing after, so $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is a local maximum. f is decreasing before $\frac{5\pi}{4}$, and increasing after, so $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$ is a local minimum.

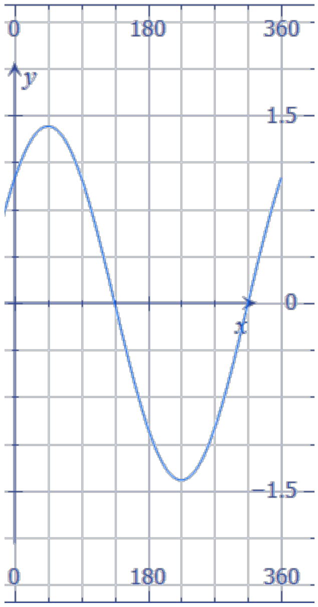
g) Inflection points: $f''(x) = 0 \rightarrow \quad -\sin x - \cos x = 0 \rightarrow \quad \sin x = -\cos x \rightarrow \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$. So the two candidates for inflection points are $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$. Next, checking for changes in concavity at each point: $f''\left(\frac{\pi}{2}\right) = -1$, so f is concave down here. $f''(\pi) = 1$, so f is concave up here. $f''\left(\frac{11\pi}{6}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}$, so f is concave down here. So f changes concavity at both $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$, and these are our two inflection points.

h) Intervals of concavity: Now, we generalize what we found in part g):

f is concave down from $\left(-\infty, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, \infty\right)$
f is concave up from $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$.

(We also note here that since $f''\left(\frac{\pi}{4}\right)$ is concave down, it shows $\left(\frac{\pi}{4}, \sqrt{2}\right)$is a local maximum, and since $f''\left(\frac{5\pi}{4}\right)$ is concave up, it shows $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$ is a local minimum. It is another way to find part f)).

i) Sketch:



EXAMPLE:

Let $f(x) = \frac{1}{x}$. Let us again sketch the curve by finding the steps above:

a) x and y intercepts: x-intercept(s): $\frac{1}{x} = 0 \rightarrow \quad 1 = 0$, so no x-intercepts. y-intercept: $\frac{1}{0}$ is undefined, so no y-intercept.

b) Vertical and horizontal asymptotes: Vertical asymptote: Set the denominator equal to zero: $x = 0$. Since it does not cancel a factor in the numerator, it is our only vertical asymptote. Horizontal asymptote: $\lim_{x \to \pm\infty} \frac{1}{x} = 0$, so $y = 0$ is our horizontal asymptote.

c) 1st and 2nd derivatives: $f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}$.

d) Critical points: We set $f'(x) = 0 \rightarrow \quad -\frac{1}{x^2} = 0 \rightarrow \quad -1 = 0$, so there are no places where f has a horizontal tangent. There is a value where $f'(x)$ does not exist, but it is not in the domain of f, so no critical points.

e) Intervals of increase or decrease: The only place we have to check is on either side of the vertical asymptote, where f is undefined: $f'(-1) = -1$ so f is decreasing here. $f'(1) = -1$, so f is also decreasing here.

So f is decreasing from $(-\infty, 0) \cup (0, \infty)$.

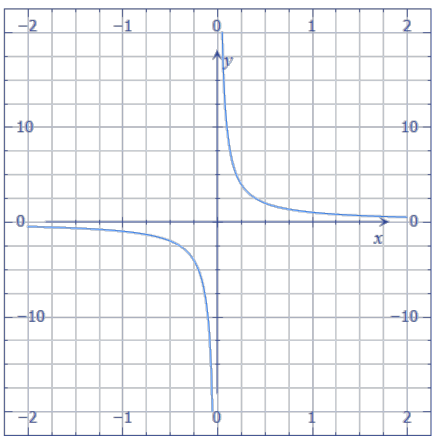
f) Local Extrema: There are no local extrema, since there are no critical points (recall local extrema can only occur at critical points).

g) Inflection points: $\frac{2}{x^3} = 0 \rightarrow \quad 2 = 0$, so there are no inflection points such that $f''(x) = 0$. At $x = 0$, $f''(x)$ does not exist, but it is also not in the domain of f. So there are no inflection points.

h) Intervals of concavity: There are no inflection points, but f could change concavity at undefined values, so we check on either side of $x = 0$. $f''(-1) = -2$, so f is concave down here. $f''(1) = 2$, so f is concave up here.

f is concave down from $(-\infty, 0)$
f is concave up from $(0, \infty)$.

i) Sketch:



EXAMPLE:

Let $f(x) = \frac{1}{x^2-1}$. Let us again sketch the curve by finding the steps above:

a) x and y intercepts: x-intercept(s): $\frac{1}{x^2-1} = 0 \rightarrow \quad 1 = 0$, so no x-intercepts. y-intercept: $y = \frac{1}{-1} = -1$.

b) Vertical and horizontal asymptotes: Vertical asymptote: Set the denominator equal to zero: $x^2 - 1 = 0 \rightarrow \quad x = \pm 1$. Since it does not cancel a factor in the numerator, our vertical asymptotes are $x = \pm 1$. . Horizontal asymptote: $\lim_{x\to\pm\infty} \frac{1}{x^2-1} = 0$, so $y = 0$ is our horizontal asymptote.

c) 1st and 2nd derivatives: $f(x) = (x^2 - 1)^{-1} \rightarrow$

$$f'(x) = -(x^2 - 1)^{-2} \cdot 2x = \frac{-2x}{(x^2 - 1)^2}.$$

$$f''(x) = \frac{(x^2 - 1)^2(-2) - 2(x^2 - 1) \cdot (2x) \cdot (-2x)}{(x^2 - 1)^4} = \frac{-2(x^2 - 1) + 8x^2}{(x^2 - 1)^3} = \frac{6x^2 + 2}{(x^2 - 1)^3}$$

d) Critical points: We set $f'(x) = 0 \rightarrow \frac{-2x}{(x^2-1)^2} = 0 \rightarrow 2x = 0 \rightarrow x = 0$. So we have a critical point at $(0, -1)$. At $x = \pm 1$, $f'(x)$ does not exist, but neither are in the domain of f. So $(0, -1)$ is our only critical point.

e) Intervals of increase or decrease: We need to check on either side of our critical point, as well as on either side of both undefined values (in this case, our vertical asymptotes). $f'(-2) =$ positive, so f is increasing here. $f'\left(-\frac{1}{2}\right) =$ also positive. $f'\left(\frac{1}{2}\right) =$ negative, so f is decreasing here. $f'(2) =$ also negative.

f is increasing from $(-\infty, -1) \cup (-1,0)$
f is decreasing from $(0,1) \cup (1, \infty)$.

(Note: We do NOT say f increasing from $(-\infty, 0)$ because f is undefined at $x = -1$. Similarly, for f decreasing.)

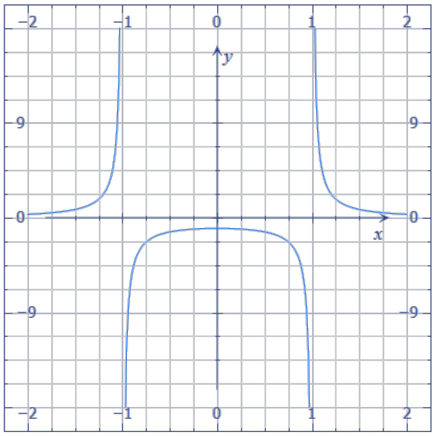
f) Local Extrema: $f'(x)$ is increasing before $x = 0$, and decreasing after, so $(0, -1)$ is a local maximum.

g) Inflection points: $f''(x) = 0 \rightarrow \frac{6x^2+2}{(x^2-1)^3} \rightarrow 6x^2 = -2$ means no inflection points. $f''(x)$ does not exist at $x = \pm 1$, but they are not in the domain of f.

h) Intervals of concavity: We have no inflection points, but we need to check for changes in concavity around the undefined values (vertical asymptotes): $f''(-2) =$ positive, so f is concave up, $f''(0) =$ negative, so f is concave down, and $f''(2) =$ positive, and f is concave up again.

f is concave up $(-\infty, -1) \cup (1, \infty)$
f is concave down $(-1,1)$.

i) Sketch:



EXAMPLE:

Let $f(x) = x^{\frac{2}{3}}$. Let us again sketch the curve by finding the steps above:

a) x and y intercepts: x-intercept(s): $(0,0)$ is the only x and y-intercept.

b) Vertical and horizontal asymptotes: There are no vertical or horizontal asymptotes.

c) 1st and 2nd derivatives: $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{\frac{1}{3}}}$, $f''(x) = -\frac{2}{9}x^{-\frac{4}{3}} = -\frac{2}{9x^{\frac{4}{3}}}$.

d) Critical points: We set $f'(x) = 0 \rightarrow \frac{2}{3x^{\frac{1}{3}}} = 0 \rightarrow 2 = 0$ so there are no critical points for which f has a horizontal tangent line. However, when $x = 0$, it is a value for which $f'(x)$ does not exist, and this one is in the domain of f. Therefore, $(0,0)$ is a critical point of f.

e) Intervals of increase or decrease: We check on either side of our critical point $(0,0)$. $f'(-1) = -\frac{2}{3}$, so f is decreasing, and $f'(1) = \frac{2}{3}$ so f is increasing.

f is decreasing from $(-\infty, 0)$
f is increasing from $(0, \infty)$.

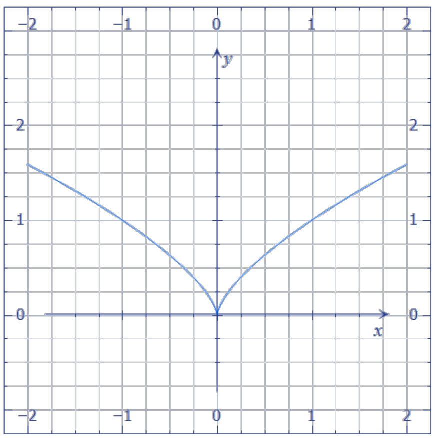
f) Local Extrema: $f'(x)$ is decreasing before $x = 0$, and increasing after, so $(0, -1)$ is a local minimum.

g) Inflection points: $f''(x) = 0 \rightarrow \quad -\frac{2}{9x^{\frac{4}{3}}} = 0 \rightarrow \quad -2 = 0$ which does not give an inflection point. However at $x = 0, \quad f''(x)$ does not exist, and is in the domain of f. Therefore, we will check for concavity changes on either side of 0: We notice that $f''(x)$ is always negative on both sides of $x = 0$. Therefore, there are no inflection points.

h) Intervals of concavity: We notice from part g), that:

f is concave down $(-\infty, 0) \cup (0, \infty)$.

i) Sketch:



SLANT ASYMPTOTE EXAMPLE:

Let $f(x) = \frac{x^2-9}{x-1}$. Let us again sketch the curve by finding the steps above:

a) x and y intercepts: x-intercept(s): $\frac{x^2-9}{x-1} = 0 \rightarrow \quad x = \pm 3$. To find the y-intercept: $\frac{0^2-9}{0-1} = 9$.

b) Asymptotes:

Vertical Asymptote: $x - 1 = 0 \rightarrow \quad x = 1$. (Reminder: Does not cancel a factor in the numerator, or it would be a hole)

There are no horizontal asymptotes for this one. ($\lim_{x\to\infty} \frac{x^2-9}{x-1} = \infty, \quad \lim_{x\to-\infty} \frac{x^2-9}{x-1} = -\infty$). Instead, there is a slant asymptote. How do we know? When the degree of the numerator is one degree greater than the degree of the denominator, instead of a horizontal asymptote, we have a slant asymptote. (Note: if the degree of the numerator is more than one degree greater than the denominator, we will not have a horizontal nor a slant asymptote). (We may still have vertical asymptote(s)).

Okay, so how do we find the slant asymptote? We perform long division, $\frac{x^2-9}{x-1} = x + 1 - \frac{8}{x-1.}$. This suggests that $y = x + 1$ is our slant asymptote. $f(x) - (x + 1) = -\frac{8}{x-1}$. $\lim_{x\to\pm\infty} -\frac{8}{x-1} = 0$ $\rightarrow$ $f(x) - (x + 1) = 0 \rightarrow$ $f(x) = x + 1 \rightarrow$ $y = x + 1$ is our slant asymptote. (Note that when the degree of the numerator is one degree greater than the degree of the denominator, we always get a linear equation performing long division, and after taking the limit).

To Summarize: The vertical asymptote is $x = 1$, and the slant asymptote is $y = x + 1$.

c) 1st and 2nd derivatives: $f'(x) = \frac{(x-1)\cdot 2x-(x^2-9)}{(x-1)^2} = \frac{\boldsymbol{x^2-2x+9}}{\boldsymbol{(x-1)^2}}$,

$$f''(x) = \frac{(x-1)^2 \cdot (2x-2) - 2(x-1)\cdot(x^2-2x+9)}{(x-1)^4}$$
$$= \frac{(2x-2)\cdot[(x^2-2x+1)-(x^2-2x+9)]}{(x-1)^4} = \frac{(2x-2)\cdot(-8)}{(x-1)^4}$$
$$= \frac{-16x+16}{(x-1)^4} = -\frac{16(x-1)}{(x-1)^4} = -\frac{\mathbf{16}}{\boldsymbol{(x-1)^3}}.$$

d) Critical points: We set $f'(x) = 0 \rightarrow$ $\frac{x^2-2x+9}{(x-1)^2} = 0 \rightarrow$ $x^2 - 2x + 9 = 0 \rightarrow$ $x = \frac{2\pm\sqrt{4-4(9)}}{2}$ has no solutions, and $x = 1$ is not in the domain of f, so no critical points.

e) Intervals of increase or decrease: Since we have no critical points, we need only check on either side of the vertical asymptote: $f'(0) =$ positive, so f is increasing, and $f'(2) =$ also positive, so f is increasing from $(-\infty, 1) \cup (1, \infty)$.

f) Local Extrema: f has no local extrema, since there are no critical points.

g) Inflection points: $f''(x) = 0 \rightarrow$ $-\frac{16}{(x-1)^3} = 0 \rightarrow$ $-16 = 0$, so no inflection points. (Note that $x = 1$ is not in the domain of f.

h) Intervals of concavity: We have no inflection points, but we need to check for changes in concavity around the undefined value (vertical asymptote): $f''(0) =$ positive, so f is concave up here, and $f''(2) =$ negative so f is concave down here.

f is concave up from $(-\infty, 1)$
f is concave down from $(1, \infty)$.

i) Sketch:

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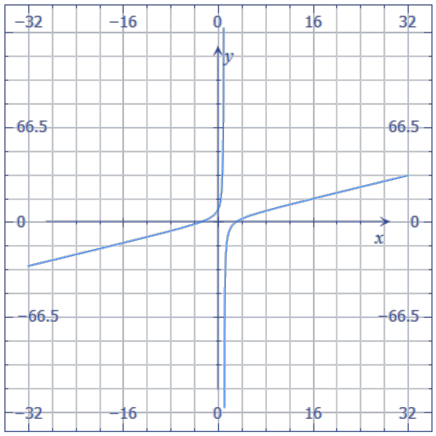


EXERCISES:

Use the following steps to sketch the graph of the function:

a) x and y intercepts (x-intercepts for all, except polynomials over degree 2).
b) Vertical and horizontal asymptotes
c) 1^{st} and 2^{nd} derivatives
d) Critical points
e) Intervals of Increase or decrease
f) Local/Relative extrema
g) Inflection points
h) Intervals of concavity
i) Sketch

1) $f(x) = 2x^2 + 4x + 2$:

2) $f(x) = x^2 - 5x + 6$:

3) $f(x) = x^2 - 16$:

4) $f(x) = x^3 + 3x^2 + 3x$:

5) $f(x) = x^3 + 9x^2$:

6) $f(x) = x^3 - 3x^2 + 1$:

7) $f(x) = -x^3 + 3x - 2$:

8) $f(x) = \frac{1}{3}x^3 - 2x - 3$

9) $f(x) = x^3 - 6x^2 + 9x + 1$:

10) $f(x) = x^4 + 4x^2$:

11) $f(x) = x^4 + 14x^2 + 48x$:

12) $f(x) = x^4 - 2x^3$:

13) $f(x) = \cos x - \sin x,\ [0,2\pi]$:

14) $f(\theta) = \sin 2\theta + \cos 2\theta,\ [0,\pi]$:

15) $f(t) = \sin^2 t,\ [0,2\pi]$:

16) $f(x) = \sqrt{3}\sin x - \cos x, \quad [0,2\pi]$:

17) $f(x) = \frac{\cos x}{1-\sin x}$:

18) $f(x) = \frac{3}{x}$:

19) $f(x) = -\frac{2}{x}$:

20) $f(x) = \frac{3}{x-1}$:

21) $f(x) = \frac{2}{2x-2}$:

22) $f(x) = \frac{x-1}{x}$:

23) $f(x) = -\frac{5}{x+3}$:

24) $f(x) = \frac{1}{x^2}$:

25) $f(x) = \frac{1}{x^2-4}$:

26) $f(x) = -\frac{2}{x^2+3}$:

27) $f(x) = \frac{x^2}{x^2-9}$:

28) $f(t) = \frac{x^2-2}{x}$:

29) $f(x) = \frac{x^2-4}{x+2}$:

30) $f(t) = (t-1)^{\frac{1}{3}}$:

31) $f(x) = 2(x-2)^{\frac{1}{3}}$:

32) $f(x) = (x+1)^{\frac{2}{3}}$:

33) $f(x) = 3(x-3)^{\frac{2}{3}}$:

CHAPTER 3
SECTION 5
APPLICATIONS: OPTIMIZATION

We now get to something very exciting! We get another direct application of derivatives, which is a direct application of most of the ideas in this chapter!

What is optimization? It is the application of finding extrema, i.e. maximizing and minimizing functions.)

These are a form of what people like to call word problems (or story problems).

STEPS FOR OPTIMIZATION:

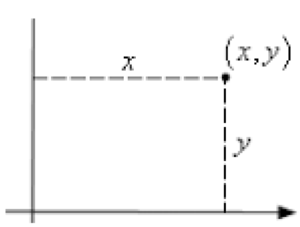
(Note: These steps come after carefully reading the problem).

1) Draw a picture, (or make a table) if plausible.
2) Label the picture; or if no picture, choose variables.
3) Write out an expression for the function that you want to optimize:
   Note: Many of these problems have more than one variable:
   If more than one variable, you will also need an equation relating the variables.
4) Get the equation in step 3) down to 1 variable, by solving for one in terms of the other. (Note, you may find other relationships to compress them all if needed).
5) Substitute the new equation in part 4) into the function you are trying to optimize from step 3).
6) Find all critical points, by solving $f' = 0$.
7) Show the critical point(s) are a maximum or a minimum by using either the first derivative test, or the second derivative test.

EXAMPLE:

Sally has a rectangular fence she wants to build. She has 200 ft. of fencing material. What are the dimensions that will maximize the area, and what is the maximum area?

1) First, we draw a picture.



2) We have also labeled the picture.

3) Here, we need two expressions:

   a) We need a function that we are trying to maximize: In this case, it is Area: $A = xy$.
   b) In this case we have 2 variables, so we need an equation the relates them: $2x + 2y = 200$.

4) Let us solve for y: $y = 100 - x$: (Note: It didn't matter which variable we chose to solve for, as either would lead to the same result).

5) We substitute 4) into our area function: $A = x(100 - x) = 100x - x^2$.

6) We find all critical points: $A' = 100 - 2x = 0 \rightarrow \quad x = 50\ ft$. When $x = 50$,

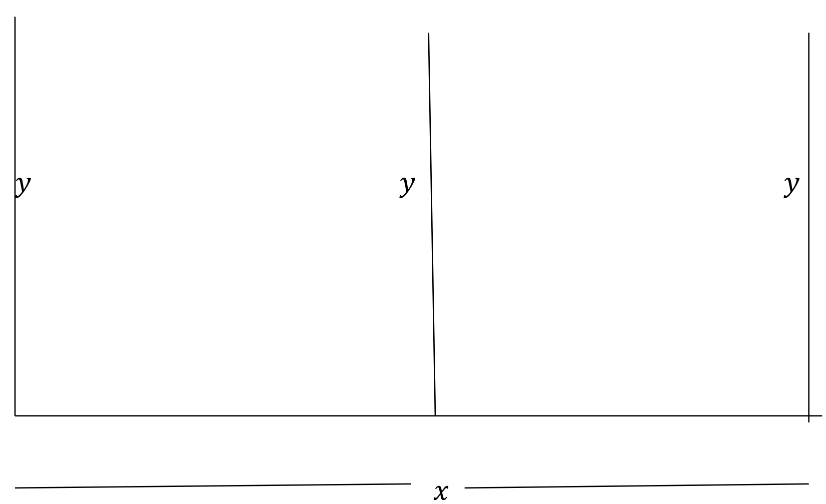
   $y = 100 - 50 = 50$. The area is $50 \times 50 = 2500\ ft^2$.

7) We will use the second derivative to show that $x = 50$ is a maximum and not a minimum: $A'' = -2$, which is always concave down. Therefore, $(50{,}50)$ is a local maximum.

EXAMPLE:

Joyce is building a fence for her two dogs. She wants to make two separate areas for them, with fencing down the middle. She will use her house as one side of the area. Joyce has 400 ft. of fencing. What are the dimensions that will maximize the area, and what is the total area that will be enclosed?

1) First, we draw a picture:



2) We have also labeled the picture. (Note: I have labeled the whole horizontal side, x, instead of $2x$. This way, we can avoid some fractions.)

3) Here, we need two expressions:

   a) We need a function that we are trying to maximize: In this case, it is Area: $A = xy$.
   b) In this case we have 2 variables, so we need an equation the relates them: $x + 3y = 400$.

4) In this case, we choose to solve for x to avoid fractions: $x = 400 - 3y$.

5) We substitute 4) into our area function: $A = (400 - 3y)y = 400y - 3y^2$.

6) We find all critical points: $A' = 400 - 6y = 0 \rightarrow \quad y = 66.66\, ft$. When $y = 66.66$,

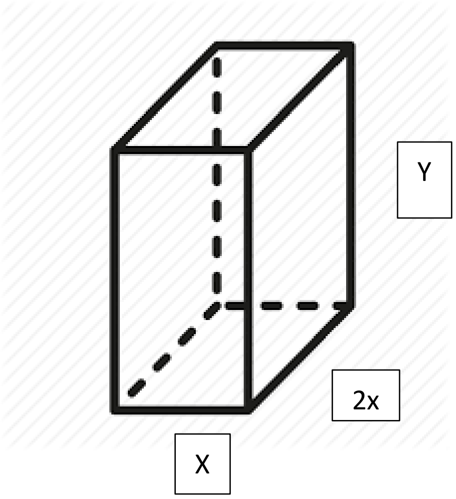
   $x = 400 - 3 \cdot 75 = 200.02\, ft$. $A = 66.66 \times 200.02 = 13{,}333.33\, ft^2$.

8) We will use the second derivative to show that $y = 75$ is a maximum and not a minimum: $A'' = -6$, which is always concave down. Therefore, $(200.02, 66.66)$ is a local maximum.

EXAMPLE:

George is building a rectangular container out of metal, with no top, to store some garden supplies. George has a fixed surface area of $150\, ft^2$. He wants to maximize the volume. The length is twice the width. What are dimensions that will maximize the volume, and what is the volume?

1) First, we draw a picture:



2) We have also labeled the picture.
3) Here, we need two expressions:

   a) We need a function that we are trying to maximize: In this case, it is Volume: $V = 2x^2y$.
   b) In this case we have 2 variables, so we need an equation the relates them:

   $$2x^2 + 2xy + 4xy = 2x^2 + 6xy = 150.$$

4) In this case we solve for y: $y = \frac{150-2x^2}{6x}$.

5) We substitute 4) into our volume function: $V = 2x^2\left(\frac{150-2x^2}{6x}\right) = \frac{1}{3}(150x - 2x^3)$.

6) We find all critical points: $V' = \frac{1}{3}(150 - 6x^2) = 50 - 2x^2 = 0 \rightarrow \quad x^2 = 25 \rightarrow \quad x = \pm 5$.

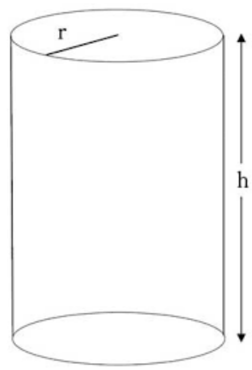
   x cannot be -5, because length cannot be negative. So $x = 5\,ft$, $y = \frac{10}{3}\,ft$. Volume is $2 \cdot 25 \cdot \frac{10}{3} = \frac{500}{3} ft^3$.

9) We will use the second derivative to show that $x = 5$ is a maximum and not a minimum: $V'' = -4x \rightarrow \quad V''(5) = -20$ which is concave down. Therefore, $\left(5, \frac{10}{3}\right)$ is a local maximum.

EXAMPLE:

Teresa has a soup company. She is trying to save money on the materials she is using for her cans. She uses cylindrical cans to contain her soup. If her can needs a volume of $35\ in^3$, what dimensions will minimize the surface area, and what is the surface area?

1) We draw a picture:



2) We have also labeled the picture.

3) Here, we need two expressions:

   a) We need a function that we are trying to minimize: In this case, it is Surface Area:
   b) $S = 2\pi r^2 + 2\pi rh$.
   c) In this case we have 2 variables, so we need an equation the relates them: $V = \pi r^2 h = 35$.

4) Here we solve for h: $h = \frac{35}{\pi r^2}$.

5) We substitute 4) into our surface area function: $S = 2\pi r^2 + 2\pi r \cdot \left(\frac{35}{\pi r^2}\right) = 2\pi r^2 + \frac{70}{r}$.

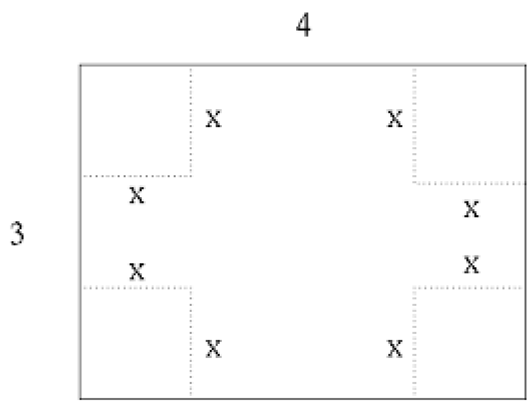
6) We find all critical points: $S' = 4\pi r - \frac{70}{r^2} = 0 \rightarrow \quad \frac{70}{r^2} = 4\pi r \rightarrow \quad 4\pi r^3 = 70 \rightarrow \quad r^3 = \frac{35}{2\pi} \rightarrow$ $r \approx 1.77. \quad h \approx 3.56.$

7) $S'' = 4\pi + \frac{40}{r^3}$ is always positive, since r is always positive. This makes S concave up whenever $r > 0$, so we have a local minimum.

EXAMPLE:

Joey is making a rectangular box out of a rectangular sheet of cardboard. The sides are $3 \times 4\, ft.$ He is cutting out the corners, so he can fold it up and tape it together. What size cut should he make to maximize the volume of the box?

1) First we draw a picture;



2) We have already labeled the picture.

3) In this case we only need one expression for Volume: $V = (3-2x)(4-2x)x =$

   $x(12 - 14x + 4x^2) = 4x^3 - 14x^2 + 12x.$
4) We can skip this
5) And this
6) We skip ahead to critical points: $V' = 12x^2 - 28x + 12 = 0 \rightarrow \quad 3x^2 - 7x + 3 = 0 \rightarrow \quad x = \frac{7 \pm \sqrt{13}}{6} \approx 1.77,\ .57$. We immediately see that we have to take $\frac{7-\sqrt{13}}{6} \approx .57$, as the other value is bigger than 3 when doubled.

7) $V'' = 24x - 28$. At $x \approx .57$, V'' is negative. So V is concave down, and a maximum.

EXAMPLE:

Find the point on the graph $y = x - 1$ that is closest to the point $(0,2)$. For this problem, we need the distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This involves a square root. We can minimize the distance squared instead of the distance to obtain the same result.

1) We do not need a picture for this.

2) Our variables are already chosen.

3) We want to minimize $d^2 = (y-2)^2 + (x-0)^2$.

4) This step is already done.

5) We substitute $y = x - 1$ for y: $(x-1-2)^2 + (x-0)^2 = (x-3)^2 + x^2$.

6) We find all critical points: $(d^2)' = 2(x-3) + 2x = 2x - 6 + 2x = 4x - 6$.
$4x - 6 = 0 \rightarrow \quad x = \frac{3}{2}$. So $\left(\frac{3}{2}, \frac{1}{2}\right)$ is our only critical point.

7) $(d^2)'' = 4$ is concave up always. So, $\left(\frac{3}{2}, \frac{1}{2}\right)$ is a local minimum.

EXAMPLE:

This next example is a business example. You should know that profit equals revenue minus cost, and revenue equals your price function times the number of items you are selling.

Jenny is opening a clothing line of dresses:

The price, p is modeled by $p = 100 - 0.5x$.

The total cost of producing x dresses is $C(x) = 1000 + 0.5x^2$.

a) Find the total revenue, $R(x)$. $R(x) = p \cdot x = 100x - 0.5x^2$.

b) Find the total profit, $P(x)$. $P(x) = R(x) - C(x) = 100x - 0.5x^2 - (1000 - 0.5x^2) =$
$100x - 1000$.

c) How many dresses must Jenny sell to maximize profit? We find the first derivative of profit, and set it equal to zero to find our critical number: $P' = 100$. Therefore we must sell 100 dresses in order to maximize our profit.

d) What is the maximum profit? $P(100) = 100 \cdot 100 - 1000 = \$9000$.

e) What price per dress must Jenny charge to maximize her profit? $p = 100 - 0.5 \cdot 100 = \$50$.

EXERCISES:

1) Of all the rectangles whose perimeter is 50 cm, find the dimensions of the rectangle with the maximum area? What is the maximum area?

2) Kevin is building a rectangular fence for his dogs. He has 200 feet of fencing. What are the dimensions that will produce the maximum area, and what is the maximum area?

3) Anna is building a fence for her two dogs. She wants to make two separate areas for them, with fencing down the middle. She will use her house as one side of the area. Joyce has 300 ft. of fencing. What are the dimensions that will maximize the area, and what is the total area that will be enclosed?

4) Joe is building a fence for his 3 dogs. He wants to keep them all separate, or they will fight. He has 600 feet of fencing. He will use the barn as one side of the area enclosed. What are the dimensions that will maximize the area, and what is the total area that will be enclosed?

5) Fred is building a rectangular fence for his garden. His garden must have an area of $12{,}000\ ft^2$ in order for his plan to have enough room. He wants to minimize the cost of the materials. What are the dimensions that will minimize the cost?

6) Lara is building a fence for her yard. One side of the yard will be her house. She needs an area of $10{,}000\ ft^2$. She has not yet purchased the fencing material. Lara wants to save on cost. If the cost of the fencing material is $20/lateral foot, what are the dimensions that will minimize the materials purchased, and how much will she need to spend on her fencing supplies?

7) Jane is building a rectangular container out of wood, including a top, to store some household items. Jane has a fixed surface area of $100\ ft^2$. She wants to maximize the volume. The length is three times the width. What are dimensions that will maximize the volume, and what is the volume?

8) A class project involves building a box out of cardboard. The class has quite a bit of cardboard for many projects. The box needs to have a volume of $30\ ft^3$. The length and the width will be equal. The class wants to use the minimum amount of cardboard necessary to obtain the desired volume. What are the dimensions that will minimize the surface area and what is the surface area?

9) Mindy has a soup company. She is trying to save money on the materials she is using for her cans. She uses cylindrical cans to contain her soup. If her can has a volume of $570\ cm^3$, what dimensions will minimize the surface are, and what is the surface area?

10) Johnny is building a cylindrical tank out of metal to hold used motor oil. He has $200\ ft^2$ of metal to be used. What are the dimensions that will maximize the volume, and what is the maximum volume?

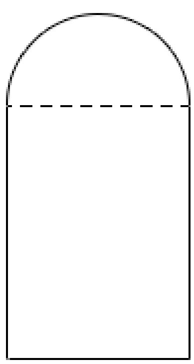
11) Bob is making a box with rectangular box out of a squarer sheet of cardboard. The sides are $2 \times 2\ ft$. He is cutting out the corners, so he can fold it up and tape it together. What size cut should he make to maximize the volume of the box?

12) Sharla is making a box with rectangular box out of a rectangular sheet of cardboard. The sides are $4 \times 5\ ft$. She is cutting out the corners, so she can fold it up and tape it together. What size cut should she make to maximize the volume of the box?

13) Enclose a rectangle within a circle. Find the maximum area of the rectangle that lies within a circle of radius 5.

14) What are the dimensions of an isosceles triangle that will produce the largest area, inscribed in a circle of radius 8.

15) Sarah is building her first house. She wants to put in a Norman window. This window is a rectangular, with a semi-circle on top. The perimeter will be 50 ft. What are the dimensions that will maximize the area, and what is the maximum area:



16) A cylinder is inscribed in a sphere of radius 10. What are the dimensions of the cylinder that will maximize its volume, and what is its volume?

17) Find the point on the graph of $y = 3x - 2$ that is closest to $(0,0)$.

18) Find the point on the graph of $y = x + 2$ that is closest to $(1,0)$.

19) Find the point t on the graph of $y = \sqrt{x}$ that is closest to $(1,0)$.

20) Find the point on the graph of $x^2 + y^2 = 1$ that is farthest from $(1,0)$.

21) A poster will have a total area of $1150\ cm^2$. If the margins are $3\ cm$ all around, what dimensions will give the largest printed area, and what is the printed area?

22) Acme Appliances are selling some brand new stoves. The price per stove is modeled by

$p = 800 - 0.5x$. The cost of producing x stoves is modeled by $C(x) = 6000 + 0.5x^2$.

a) Find the total revenue, $R(x)$.
b) Find the total profit, $P(x)$.
c) How many stoves must Acme sell in order in order to maximize its profit?
d) What is the maximum profit?
e) What price per stove must be changed to maximize profit?

23) Hunter is opening a clothing line of jeans. The price, p is modeled by $p = 150 - 0.5x$. The total cost of producing x dresses is $C(x) = 1500 + 0.5x^2$.

a) Find the total revenue, $R(x)$.
b) Find the total profit, $P(x)$.
c) How many jeans must Hunter sell in order in order to maximize his profit?
d) What is the maximum profit?
e) What price per stove must be changed to maximize profit?

CHAPTER 3
SECTION 6
INDEPTERMINATE FORMS OF LIMITS USING L'HOSPITAL'S RULE
NEWTON'S METHOD

L'HOSPITAL'S RULE:

Before we start let's talk about L'Hospital! First off, let's talk about how to pronounce it! This may seem unimportant, but we don't want to sound silly when we discuss him with all our friends and neighbors. ☺. It is French and pronounced: Low Pee Tahl. There! Not El Hospital! Okay, on to the mathematician. He was born Guillaume-François-Antoine Marquis de L'Hôpital in 1661 in Paris, France. He showed a natural aptitude early on, and had a military career. He is mostly known, because of his association with Johann Bernoulli (discussed in Chapter 0). L'Hospital contributed to calculus, including determining tangents to curves. He became a professor of mathematics at Groningen in 1695, and in 1696 he published the first textbook on differential calculus. L'Hospital's Rule was in chapter 9. (Wikipedia).

L'Hospital's Rule applies to limits: It only applies to indeterminate forms of $\frac{\infty}{\infty}$ or $\frac{0}{0}$. There are other indeterminate forms as well. We can often take the other forms, and manipulate them into one of the forms in which L'Hospital's Rule applies.

We have already dealt with these forms. For example: $\lim\limits_{x\to\infty} \frac{2x^2+2x-9}{3x^2+7}$. This is a form of $\frac{\infty}{\infty}$. Previously, we divided all terms in the numerator and denominator by x^2 to get $\lim\limits_{x\to\infty} \frac{2+\frac{2}{x}-\frac{9}{x^2}}{3+\frac{7}{x^2}} = \frac{2}{3}$. This is a case where L'Hospital's Rule would also apply. But what if we had $\lim\limits_{x\to\infty} \frac{\ln x}{x}$? This would be much harder to algebraically manipulate. This is where L'Hospital's Rule comes in. (Note: it is in the form of $\frac{\infty}{\infty}$).

L'Hospital's Rule: Let f and g be differentiable functions on an open interval I that contains a, except possibly at a. Also let $g'(x) \neq 0$. If $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ is a form of either $\frac{\infty}{\infty}$ or $\frac{0}{0}$, then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$.

We can keep applying L'Hospital's Rule over and over, as many times as we like until it no longer applies. I.E., as long as we have a form of $\frac{\infty}{\infty}$ or $\frac{0}{0}$, we can continue to apply it to the limit. Once it is no longer in one of these forms, we must stop.

EXAMPLE: Let's go back to our previous example: $\lim\limits_{x\to\infty} \frac{2x^2+2x-9}{3x^2+7}$. After some algebraic manipulation, we found the limit was $\frac{2}{3}$. Let us now use L'Hospital's Rule to calculate the limit: $\lim\limits_{x\to\infty} \frac{2x^2+2x-9}{3x^2+7} = \lim\limits_{x\to\infty} \frac{4x+2}{6x} = \lim\limits_{x\to\infty} \frac{4}{6} = \frac{2}{3}$. We observe that we got the same answer. Also note that we had to perform L'Hospital's Rule twice.

EXAMPLE:

Let's go back to the other example we mentioned previously: $\lim_{x\to\infty}\frac{\ln x}{x}$. This one proved to be quite difficult to algebraically manipulate to get the answer. In fact, we did not try. Let us now apply L'Hospital's Rule: : $\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0.$

EXAMPLE:

Let us calculate the $\lim_{x\to\infty}\frac{x^2-7x}{e^x}$. We have a form of $\frac{\infty}{\infty}$, so we apply L'Hospital's Rule: $\lim_{x\to\infty}\frac{x^2-7x}{e^x} = \lim_{x\to\infty}\frac{2x-7}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0.$

EXAMPLE:

Let's look at $\lim_{x\to 0}\frac{\cos x-1}{\sin x}$ is a form of $\frac{0}{0}$ so L'Hospital's Rule applies. $\lim_{x\to 0}\frac{\cos x-1}{\sin x} = \lim_{x\to 0}\frac{-\sin x}{\cos x} = \frac{0}{1} = 0.$

EXAMPLE:

How about $\lim_{x\to 0^+}\frac{\ln x}{\csc x}$ is a form of $-\frac{\infty}{\infty}$ which is still a form in which L'Hospital's Rule applies. It doesn't matter whether the ∞ is $\pm$. So $\lim_{x\to 0^+}\frac{\ln x}{\csc x} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\csc x\cot x} = (\text{is still a form of}\frac{\infty}{-\infty}) = -\lim_{x\to 0^+}\frac{\sin^2 x}{x\cdot\cos x} = \left(\text{Now a form of } \frac{0}{0} \text{ after some algebra}\right) = -\lim_{x\to 0^+}\frac{2\sin x\cos x}{-x\sin x+\cos x}(\text{after L'Hospital's Rule}) = -\frac{0}{1} = 0.$ (Note that we kept applying L'Hospital's Rule until we no longer had a form of $\frac{\infty}{\infty}$ or $\frac{0}{0}$.)

INDETERMINATE PRODUCTS:

Indeterminate products are a form of $0\cdot\infty$. L'Hospital's Rule does not apply to this form. So we need to rewrite $f\cdot g$ as either $\frac{f}{1/g}$ or $\frac{g}{1/f}$.

(Note: Sometimes it's easy to see which one to put on top, and sometimes it's more difficult. As in the "guess and check" method for factoring, one can always try one way; and if it gets too crazy, rewrite it the other way and try again).

EXAMPLE:

$\lim_{x\to\infty} x\,e^{-x}$ is a form of $\infty \cdot 0$, which is an indeterminate form (an indeterminate product more specifically), and we need to rewrite it in order to apply L'Hospital's Rule. This one is easy to see the rewrite $\lim_{x\to\infty} x\,e^{-x} = \lim_{x\to\infty} \frac{x}{e^x}$ is the most natural. Then, $\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$ after applying L'Hospital's Rule.

EXAMPLE:

$\lim_{x\to 0^+} x^2 \ln x$. This is a form of $0 \cdot (-\infty)$, which again needs to be rewritten. In this case we rewrite $\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}}$. (We now have a form of $-\frac{\infty}{\infty}$. Notice we rewrite the x^2 rather than $\ln x$. We do this by observing that $\frac{d}{dx}\left(\frac{1}{x^2}\right)$ will be easier to deal with than $\frac{d}{dx}\left(\frac{1}{\ln x}\right)$.) Now, $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = -\lim_{x\to 0^+} \frac{x^2}{2} = -\frac{0}{2} = 0$. (Also note we had to approach 0 from the right in keeping with the domain for $f(x) = \ln x$).

INDETERMINATE DIFFERENCES:

Indeterminate differences are a form of $\infty - \infty$. These must also be algebraically manipulated to get a form of $\frac{\infty}{\infty}$ or $\frac{0}{0}$, so that we may apply L'Hospital's Rule. (Frequently, we get a common denominator).

EXAMPLE:

$\lim_{x\to 0^+} \frac{1}{x} - \frac{2}{x^2+x}$ is a form of $\infty - \infty$. We get a common denominator so we can apply L'Hospital's Rule. $\lim_{x\to 0^+} \frac{1}{x} - \frac{2}{x^2+x} = \lim_{x\to 0^+} \frac{(x+1)-2}{x^2+x} = \lim_{x\to 0^+} \frac{x-1}{x^2+x} = \lim_{x\to 0^+} \frac{1}{2x+1}$ (Using L'Hospital's Rule) $= \frac{1}{1} = 1$.

EXAMPLE:

$\lim_{x\to 0} \cot x - \csc x$ This is a form of $\infty - \infty$. Here, we will rewrite everything in terms of sines and cosines, and then proceed to get a common denominator: $\lim_{x\to 0} \cot x - \csc x = \lim_{x\to 0} \frac{\cos x}{\sin x} - \frac{1}{\sin x} = \lim_{x\to 0} \frac{\cos x - 1}{\sin x}$ is now a form of $\frac{0}{0}$. And, we actually calculated this limit in an above example and got 0.

INDETERMINATE POWERS:

We actually have three indeterminate powers. They are the forms of:

1) 0^0
2) ∞^0
3) 1^∞

These indeterminate powers require a little more massaging (or manipulation) to get them into a form in which we can apply L'Hospital's Rule.

We must take natural logarithms of both sides in order to use rules of logarithms: After manipulating and finding the limit, we must raise the limit to the power of e to undo what we did in the beginning.

EXAMPLE:

$\lim_{x\to 0^+} x^x$ is a form of 0^0. We need to get it into a form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$. We let $y = \lim_{x\to 0^+} x^x$. We take ln of both sides to get: $\ln y = \ln \lim_{x\to 0^+} x^x = \lim_{x\to 0^+} \ln x^x$, by limit law. Then, $\lim_{x\to 0^+} \ln x^x = \lim_{x\to 0^+} x \ln x$, by a rule of logarithms. We now have an indeterminate product, which we rewrite as $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$, which is now in a form of $-\frac{\infty}{\infty}$. We can now apply L'Hospital's Rule to get: $\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to 0^+} \frac{x^2}{x} = -\lim_{x\to 0^+} x = 0$. So we calculated $\ln \lim_{x\to 0^+} x^x$. But we wanted $\lim_{x\to 0^+} x^x$. We have $\ln y = \ln \lim_{x\to 0^+} x^x = 0$. Therefore $y = \lim_{x\to 0^+} x^x = e^0 = 1$.

NEWTON'S METHOD

Newton's Method is tacked onto the end of this section, because I believe it is best taught in conjunction with a mathematical computer programming class. Doing these by hand is extremely inefficient. I placed it here merely to introduce the concept, for the completeness of this text.

Newton's Method is an approximation method to find roots of functions. For example, we are all very familiar with the quadratic equation to find roots of quadratic functions. There are other such formulas up through degree 4 (though quite cumbersome). For degree 5 or higher, there is none. We also don't have formulas for transcendental functions (like exponential, logarithmic, and trigonometric).

We must start with an approximation of x_1.

Now we need to find a formula in terms of x_1, for x_2.

We start by knowing the equation of the tangent line to $f(x)$ at x_1 is:

$$y - f(x_1) = f'(x_1)(x - x_1)$$

The x-intercept of the line is x_2, so $(x_2, 0)$ is a point on the tangent line.

Then $0 - f(x_1) = f'(x_1)(x_2 - x_1)$ is the equation of this line.

Then $(x_2 - x_1) = -\frac{f(x_1)}{f'(x_1)}$, and $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

We say x_2 is the second approximation for our root.

We keep going and get $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

To generalize: $\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$ **.**

Note that the larger n gets, the closer we get to our root.

You can now see that for any accuracy, this process will be very tedious. The ensuing examples and exercises are left to a mathematical programming class, or for a programming worksheet.

EXERCISES (For L'Hospital's Rule):

Find the following limits. Use L'Hospital's Rule whenever applicable: (You can use an easier method if it saves time)

1) $\lim\limits_{x\to\infty} \frac{2x+9}{7x-7}$

2) $\lim\limits_{x\to\infty} \frac{x}{3x-12}$

3) $\lim\limits_{x\to\infty} \frac{3x^2+2x}{x-8}$

4) $\lim\limits_{x\to\infty} \frac{x^2-9x+2}{4x^2+6x-7}$

5) $\lim\limits_{x\to\infty} \frac{4x-17}{3x^2-12x+2}$

6) $\lim\limits_{x\to\infty} \frac{2x^3-9x^2+8}{3x^3+12x-9}$

7) $\lim\limits_{x\to\infty} \frac{3x^2-9x+2}{x^3-7x^2+2x-4}$

8) $\lim\limits_{x\to\infty} \frac{x^3-19x}{x-2}$

9) $\lim\limits_{x\to 2^+} \frac{x^3-19x}{x-2}$

10) $\lim\limits_{x\to\infty} \frac{\sqrt{x^2-x-1}}{x+2}$:

11) $\lim\limits_{x\to\infty} \frac{\ln\sqrt{x}}{2x}$

12) $\lim\limits_{x\to\infty} \frac{\ln 4x}{x^3}$

13) $\lim\limits_{x\to\infty} \frac{x^2-19x}{e^{2x}}$

14) $\lim\limits_{x\to 0} \frac{e^{4x}-1}{3x}$

15) $\lim\limits_{x\to 0^-} \frac{e^x-2^x}{x^2}$

16) $\lim\limits_{x\to 0} \frac{\cos x-1}{2\sin x}$

17) $\lim_{x \to \frac{\pi}{2}} \frac{2\sin x - 2}{4\cos 4x}$

18) $\lim_{x \to \frac{\pi}{2}} \frac{x\cos x}{\sin x - 1}$

19) $\lim_{x \to \infty} \frac{\ln x^4}{x^2}$

20) $\lim_{x \to 0} \frac{2\sin 3x - \sin 5x}{4x}$

21) $\lim_{x \to \infty} x\, e^{-2x}$

22) $\lim_{x \to \infty} x^2 e^{-x}$

23) $\lim_{x \to \infty} x \sin\frac{2}{x}$

24) $\lim_{x \to 0^+} x \ln x$

25) $\lim_{x \to 0^+} 2x^3 \ln x^2$

26) $\lim_{x \to \infty} x^{\frac{1}{3}}\, e^{-x^2}$

27) $\lim_{x \to 0^+} \ln x \sin 2x$

28) $\lim_{x \to 0^+} \ln x^2 \tan x$

29) $\lim_{x \to 0^+} \frac{3}{x^2} - \frac{1}{x+3}$

30) $\lim_{x \to 0^+} \frac{2}{x} - \frac{1}{x^3+x}$

31) $\lim_{x \to 0^+} \frac{1}{x} - \frac{2}{2x^2+x}$

32) $\lim_{x \to 0} 2\cot x - \csc 2x$

33) $\lim_{x \to 1^+} \frac{3}{x^2-1} - \frac{2}{\ln x}$

34) $\lim_{x \to 0} \frac{2}{e^{2x}-1} - \frac{1}{3x}$

35) $\lim_{x \to 0^+} x^{2x}$

36) $\lim_{x \to 0^+} x^{x^2}$

37) $\lim_{x \to 0^+} (1 + \sin x)^{\frac{1}{x}}$

38) $\lim_{x \to \infty} x^{\frac{2}{x}}$

39) $\lim_{x \to \infty} 2x^{e^{-2x}}$

CHAPTER 3
SECTION 7
ANTIDERIVATIVES:

What is an antiderivative? Well, if you think of the word "anti", you usually think of its opposite. We sometimes call an antiderivative a "backwards derivative", meaning you do the opposite thing.

Let's look at this: What is $\frac{d}{dx}(x^2)$? It is $2x$. How about $\frac{d}{dx}(x^2+2)$? It is $2x$. What about $\frac{d}{dx}(x^2+107)$? It is $2x$. How about $\frac{d}{dx}(x^2+\pi)$? Again, $2x$. How about $\frac{d}{dx}(x^2+C)$, where C is any constant? Why, it is still $2x$. So what do you think the antiderivative of $f(x)=2x$ would be? Do you see it is x^2+C? I hope you can!

DEFINITION:

A function F is called an antiderivative of f on an interval I if $F'(x)=f(x)$ for all x in I.

We recall from section 3.2, The Mean Value Theorem, that if 2 functions have the same derivatives on I, that they differ by a constant. So, if F and G are any 2 antiderivatives of f, then
$F'(x)=f(x)=G'(x) \rightarrow \quad F(x)-G(x)=C \rightarrow \quad F(x)=G(x)+C.$

THEOREM:

Let F be an antiderivative of f on I, then the most general antiderivative of f on I is $\boldsymbol{F(x)+C}$**,** where C is an arbitrary constant.

POWER RULE:

Let's start with the power rule. Recall the derivative: $\frac{d}{dx}(x^n)=nx^{n-1}$. What did we do? We multiplied x by the current exponent, n, then we subtracted a 1 from the exponent. What is the opposite of subtraction? Addition. What is the opposite of multiplication? Division. So to take the antiderivative of $f(x)=x^n$, we get $\boldsymbol{F(x)=\frac{x^{n+1}}{n+1}}$. Observe that we added 1 to the exponent, and divided by the new exponent.

EXAMPLE:

Let's find the antiderivatives of the following functions: (Some will use the power rule that we just derived. Some will just be the opposite of the derivative that we know).

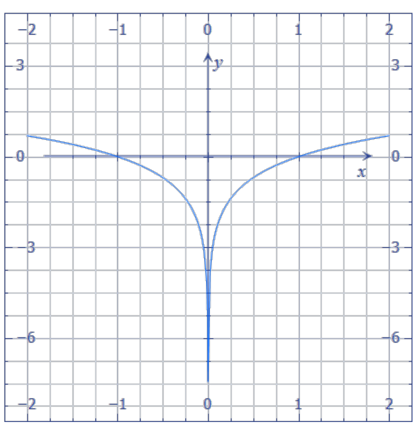
1) $f(x) = x^{\frac{1}{3}}$. Adding 1 to the exponent and dividing by the new exponent, we get

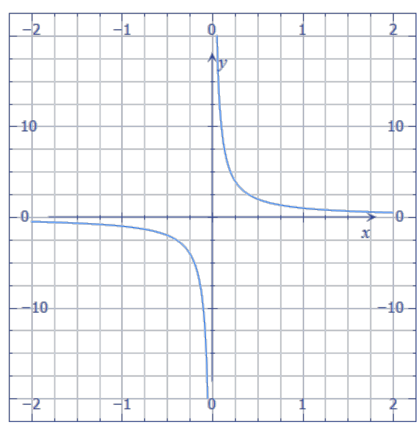
$$F(x) = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{3}{4}x^{\frac{4}{3}} + C.$$

2) $f(x) = \cos x$. No power rule here. But we know that $\frac{d}{dx}(\sin x) = \cos x$. Therefore, the antiderivative of $f(x) = \cos x$ must be $F(x) = \sin x + C$.

3) $f(x) = \frac{1}{x}$. Let's try to apply the power rule here. (You might be thinking wait! $\frac{d}{dx}(\ln x) = \frac{1}{x}$. This is true, and we will get to it in a minute. Many students will always try to apply the power rule anyway, because they forget this fact). If we were to try to apply the power rule to $f(x) = \frac{1}{x}$, we first rewrite it as $f(x) = x^{-1}$. Then adding 1 to the exponent and dividing it out would give us $F(x) = \frac{x^0}{0} + C$. Oops, we all know we can't divide by zero. Our function is undefined there. This will hopefully be a big enough red flag, as to jolt us into recalling that this particular power of x has a special rule, and remember the fact $\frac{d}{dx}(\ln x) = \frac{1}{x}$. Therefore, for $f(x) = \frac{1}{x} \rightarrow$ $F(x) = \ln|x| + C$. Okay, you might now ask, why the absolute value signs? Because the domain of $f(x) = \ln x$ is $(0, \infty)$, and the domain of $f(x) = \frac{1}{x}$ is $(-\infty, 0) \cup (0, \infty)$. We want the domains to match up, so we have an antiderivative with the same domain as $f(x) = \frac{1}{x}$. We see that $F(x) = \ln|x|$ also has domain $(-\infty, 0) \cup (0, \infty)$, which you can observe in the graph below and left:

Also observe the graph of $f(x) = \frac{1}{x}$ on the right (the derivative graph), and notice the symmetry, in how the left hand side is the negative of the right hand side. (I.e., observe how the derivative graph is correct for $f(x)$, showing where f is increasing and decreasing).





So $F(x) = \ln|x| + C$ gives us the correct antiderivative for the entire domain of $f(x) = \frac{1}{x}$.

4) $f(x) = e^x \to \quad F(x) = e^x + C$. We note that since $f(x) = e^x$ is its own derivative, it is also its own antiderivative, plus a constant.

5) $f(x) = a^x \to \quad F(x) = \frac{a^x}{\ln a} + C$. (We note that since $\frac{d}{dx}(a^x) = a^x \cdot \ln a$, and that since division is the opposite of multiplication, that we divide out the $\ln a$ instead of multiplying it.)

Let us now construct a table of antiderivatives:

TABLE OF ANTIDERIVATIVES:

Function	Antiderivative
$x^n, \;\; n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
e^x	$e^x + C$
a^x	$\frac{a^x}{\ln a} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$
$\frac{1}{\sqrt{1-x^2}}$	$sin^{-1} x + C$
$\frac{1}{1+x^2}$	$\tan^{-1} x + C$
$\cosh x$	$\sinh x + C$
$\sinh x$	$\cosh x + C$
$cf(x)$	$cF(x) + C$
$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

In the next examples, we will first find the general antiderivative (as we just practiced); but with additional conditions, we will be able to find the particular C in each case.

EXAMPLE:

1) Find f given $f'(x) = x^2 - 2x + 4$, and given that $f(0) = 3$.

   First, we find $f(x) = \frac{x^3}{3} - x^2 + 4x + C$: How do we find C? We substitute $f(0) = 3$, and algebraically solve for C: $\frac{0^3}{3} - 0^2 - 4 \cdot 0 + C = 3 \to \quad C = 3$. Therefore,

$f(x) = \frac{x^3}{3} - x^2 + 4x + 3.$

2) Find f given $f''(x) = \sin x + e^x$, and given that $f'(0) = 1,\ f(0) = 0.$

   First, we find $f'(x) = -\cos x + e^x + C$. Now to find C:
   $-\cos 0 + e^0 + C = 1 \rightarrow \quad -1 + 1 + C = 0 \rightarrow \quad C = 0$. Therefore, $f'(x) = -\cos x + e^x$.
   Next we find $f(x)$. $\ f(x) = -\sin x + e^x + C$. Next, we find C: $-\sin 0 + e^0 + C = 0 \rightarrow \quad C = 0$
   Therefore, $f(x) = -\sin x + e^x$.

3) Find f given $f'(x) = x^{\frac{1}{3}} - \frac{2}{x} + 4$, and given that $f(1) = 2$.

   First, we find $f(x) = \frac{3}{4}x^{\frac{4}{3}} - 2\ln|x| + 4x + C$. Now to find C: $\ \frac{3}{4} \cdot 1^{\frac{4}{3}} - 2\ln 1 + 4 \cdot 1 + C = 2 \rightarrow$
   $\frac{3}{4} - 0 + 4 + C = 2 \rightarrow \quad C = -\frac{11}{4}$. Therefore, $f(x) = \frac{3}{4}x^{\frac{4}{3}} - 2\ln|x| + 4x - \frac{11}{4}$.

PHYSICS:

Next, we will derive the equations of motion for Physics, oftentimes referred to as the Kinematic Equations:

We must first recall that $v(t) = s'(t)$, and that $a(t) = v'(t) = s''(t)$, where $s(t)$ is distance, $v(t)$ is velocity, and $a(t)$ is acceleration; and t is the variable for time.

We will now basically just go backwards.

We begin with acceleration as a constant. The constant acceleration to which we are referring is gravity, which we will label g. (g is 9.8 m/s).

$$\boldsymbol{a(t) = g}$$

Next, we take the antiderivative of $a(t)$ to get $v(t) = g \cdot t + C$. Let us rename C as v_0, a constant which will be our initial velocity.

$$\boldsymbol{v(t) = gt + v_0}$$

Next, we take the antiderivative of $v(t)$ to get $s(t) = \frac{gt^2}{2} + v_0 t + C$. This time we will rename C as s_0, a constant that is our startingl distance at time $t = 0$.

$$\boldsymbol{s(t) = \frac{1}{2}gt^2 + v_0 t + s_0}.$$

To summarize:

$$\boldsymbol{a(t) = g}$$
$$\boldsymbol{v(t) = gt + v_0}$$
$$\boldsymbol{s(t) = \frac{1}{2}gt^2 + v_0 t + s_0}$$

EXAMPLE:

A particle is accelerating at a rate of $a(t) = 2t - 1$. Find functions for both the velocity and position when $v(0) = 3$, and $s(0) = 2$.

$v(t) = t^2 - t + C$. Since $v(0) = 3,\ C = 3$.
$v(t) = t^2 - t + 3$

$s(t) = \frac{t^3}{3} + \frac{t^2}{2} + 3t + C$. Since $s(0) = 2,\ C = 2$.
$$s(t) = \frac{t^3}{3} + \frac{t^2}{2} + 3t + 2.$$

EXERCISES:

Find the general antiderivative for the following functions:

1) $f(x) = x - 9$
2) $f(x) = 3x^2 + 2x + 7$
3) $f(x) = 2x^2 - 5x$
4) $f(x) = x^3 + 4x^2 - 10x + 2$
5) $f(x) = x^4 + 3x^3 + \frac{1}{2}x^2$
6) $f(x) = \sqrt{2}x^3 - x^{\frac{1}{3}} - \frac{2}{x}$
7) $f(x) = \sec^2 x - 3e^x + 7$
8) $f(x) = \frac{4}{x^3} - \sqrt[3]{x^4} - 12x$
9) $f(x) = \sec x \tan x + \frac{1}{2}e^x$
10) $f(x) = \cos x + 4\sin x - \frac{1}{2x}$
11) $f(x) = \frac{4x^2 - x^{-1} + x}{x}$
12) $f(x) = \frac{\sqrt{x} - 2x + x^{-3}}{x^2}$
13) $f(x) = \frac{1}{\sqrt{1-x^2}} - \frac{3}{1+x^2}$
14) $f(x) = (x-2)(x+5)$
15) $f(x) = (x^2+3)(x-7)$
16) $f(x) = 3e^x - \frac{4}{x} + \frac{1}{1+x^2}$
17) $f(x) = x^{\frac{2}{3}} - \frac{3}{x^{\frac{1}{3}}} + 15$

Find f:

18) $f'(x) = x + 2,\;\; f(0) = 2.$

19) $f'(x) = 3x^2 - 12x + 7,\;\; f(1) = 1$

20) $f'(x) = x^3 - 6x^2 + 3x,\;\; f(0) = 4$

21) $f'(x) = 5x^4 + 9x^2 + 2x,\;\; f(1) = 2$

22) $f'(x) = \sin x - \cos x,\;\; f(0) = 1$

23) $f'(x) = e^x + x^2,\;\; f(0) = 2$

24) $f'(x) = e^x - \cos x,\;\; f(0) = 3$

25) $f'(x) = \sec^2 x - 2e^x,\;\; f(0) = 0$

26) $f'(x) = \sec x \tan x - 2x,\;\; f(0) = 4$

27) $f''(x) = x - 7,\;\; f'(1) = 1,\;\; f(0) = 2$

28) $f''(x) = x^2 - 2x + 1,\;\; f'(0) = 2,\;\; f(1) = 1$

29) $f''(x) = 4x^3 + \sqrt{x} - x^3,\;\; f'(0) = 0,\;\; f''(0) = 0$

30) $f''(x) = (x + 1)(x - 2),\;\; f'(0) = 1,\;\; f''(1) = 0$

31) $f'(x) = \frac{x^2 - 2}{x},\;\; f'(1) = 0$

32) $f''(x) = \frac{1}{1 + x^2} - \cos x,\;\; f'(0) = 0,\;\; f(0) = 1$

33) $f''(x) = x^{\frac{1}{3}} - \frac{1}{x^2} - 2x,\;\; f'(1) = 0,\;\; f(1) = 0$

34) A particle is accelerating at a rate of $a(t) = t + 1$. Find functions for both the velocity and position when $v(0) = 2$, and $s(0) = 5$.

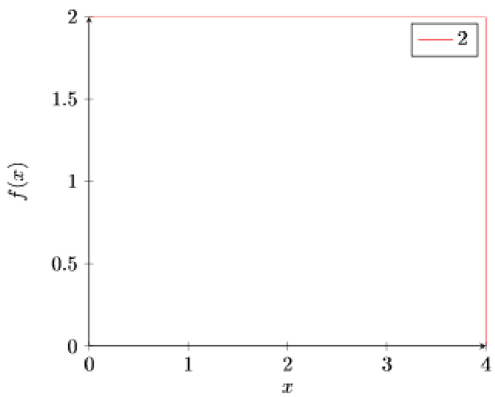
35) A particle is accelerating at a rate of $a(t) = t^2 - 1$. Find functions for both the velocity and position when $v(0) = 0$, and $s(0) = 10$.

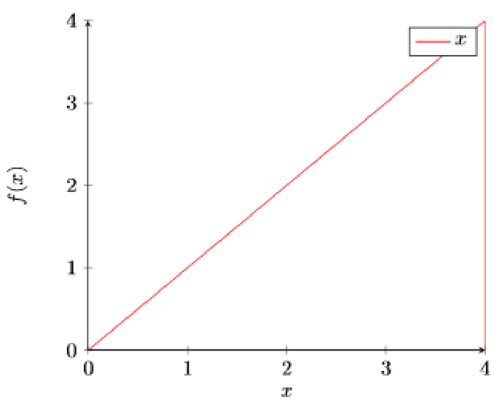
CHAPTER 4
SECTION 1
AREA APPROXIMATION

In this section, we discover how to find the area below a curve, and above the x-axis.

We begin with some familiar examples, and these were also discussed in Chapter 0:

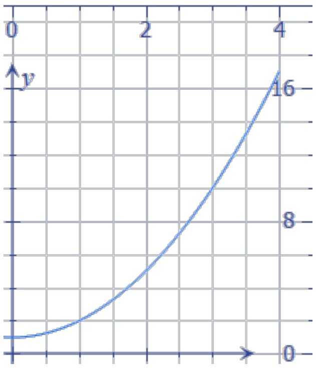
We can easily find the area under the curve for functions like $y = 2$, which is simply the area of a rectangle, bh; or for a function like $y = x$, which would be the area of a triangle, $\frac{1}{2}bh$. See the graphs below.



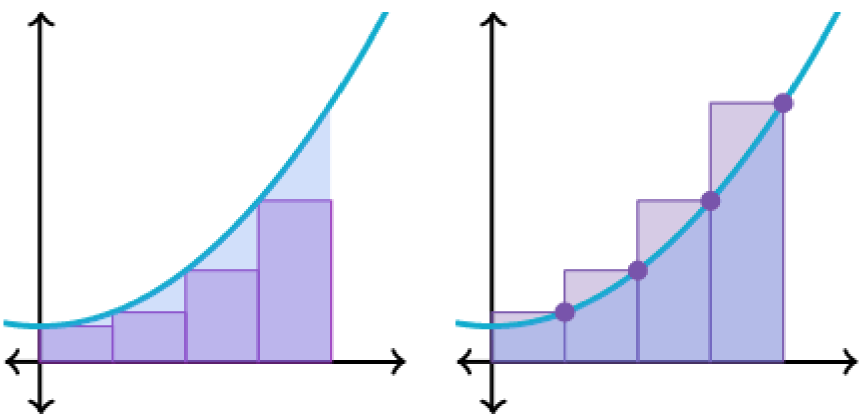


We can easily calculate the area under the curve (and above the x-axis) of these functions using basic geometry.

But now let's look at another fairly simple function: $f(x) = x^2 + 1$ from $x = 0$ to $x = 4$:



We recall from Chapter 0, that we briefly discussed how difficult the process might be. We just observed above, how easy it was to find the area of a rectangle. What if we **approximated** the area of this function instead of finding its precise area. We could use rectangles to approximate it.



We can see by the pictures above how this could be accomplished. We notice that the graph on the left would give an underestimate of the area, and the graph on the right would give an overestimate.

It is also apparent that by using only 4 rectangles, we will get a terrible approximation. But what if we had 100,000 rectangles? This would be an excellent approximation, while being a calculational nightmare.

Let us go ahead and approximate the area under the curve by calculating the area of the rectangles in each graph:

First, we recall the area of a rectangle is width times height. Here each width is the same, but each height is different. The width will be $\Delta x = \frac{b-a}{n}$, where $[a, b]$ is the interval in the x-direction, and n is the number of rectangles. $f(x_i)$ is the height of each rectangle, where x_i is the ith x-value that touches the graph. (i is arbitrary).

So the approximate area under the curve and above the x-axis is: $\sum_{i=1}^{n} f(x_i)\Delta x$. This formula is called a Riemann Sum, named after Georg Friedrich Bernhard *Riemann,* who lived 1826-1866, and was a German mathematician who made contributions to analysis, number theory, and differential geometry. (Note, technically x_i should be denoted as x_i^*, but I find this notation too cumbersome for most students. It implies that each x-value can touch the graph anywhere within the rectangle including its endpoints).

This formula may look very novel and unusual! Let's break it all down.

First, Σ simply means "sum"! It's an abbreviation for writing out each term of a sum (very convenient, especially if the sum has many terms). i is an index. It goes from 1 to n, i.e. 1,2,3,…,n. x_i is an arbitrary x-value in which its rectangle touches the graph. $f(x_i)$ is the height of the "ith" rectangle. Δx is the width of each rectangle. For our purposes, they will all be equal, and as mentioned previously, $\Delta x = \frac{b-a}{n}$.

Let us now use this formula to approximate the area of our example:

$f(x) = x^2 + 1$ from $x = 0$ to $x = 4$, using the graph on the left, i.e. using left endpoints:

We observe that the area is approximated using 4 rectangles, so $n = 4$ here. $\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$. So now we have:

$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{4} f(x_i) \cdot 1$. Now let's write out all our terms: First we find our x-values:

$x_1 = 0,\ x_2 = 1,\ x_3 = 2,\ x_4 = 3$. How did we find these? x_1 is the first x-value in which the function touches the graph. This occurs at $x = 0$. The second one occurs at $x = 1$, etc. There are 4 rectangles, and there are 4 x-values here. Note that we do not have an x-value $= 4$. (This is a common student error). We have only 4 rectangles, starting with $x = 0$. (Hint, for the graph on the right, we will have an $x = 4$, but no $x = 0$. Again, giving us 4 rectangles).

So $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{4} f(x_i) \cdot 1 =$

$[f(0) + f(1) + f(2) + f(3)] \cdot 1 =$

$[(0^2 + 1) + (1^2 + 1) + (2^2 + 1) + (3^2 + 1)] \cdot 1 =$

$1 + 2 + 5 + 10 = 18$. Recall this will be an underestimate of the area.

Next, we use the graph on the right using right endpoints. We still have 4 rectangles, so our Δx remains the same. Now $x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4$. And:

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{4} f(x_i) \cdot 1 =$$

$[f(1) + f(2) + f(3) + f(4)] \cdot 1 =$

$[(1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1)] \cdot 1 =$

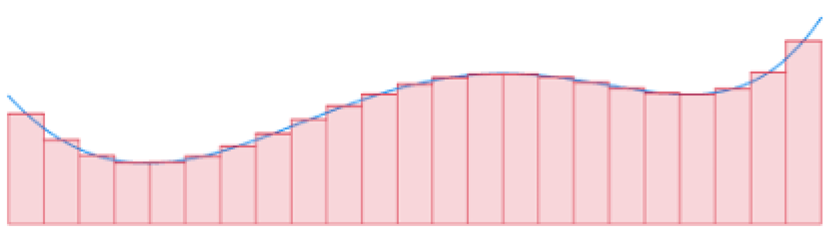
$2 + 5 + 10 + 17 = 34$. Recall this will be an overestimate of the area.

Observe that these are terrible approximations, but they serve to get the general ideas across without having to do brutal calculations by hand. Like Newton's Method, if approximations are needed, they are best done on computer software where hundreds of thousands of calculations can be done in an instant.

To Summarize: We approximated the area under a curve of a function over an interval, and above the x-axis, using rectangles. We first used left endpoints, and then we recalculated using right endpoints. We could have used the middle of each rectangle, instead of the left or right-hand corner, and this method is called the Midpoint Rule. This generally gives a better approximation. We also have a method called the Trapezoid Rule, which uses trapezoids in lieu of rectangles to get an even better approximation, or Simpson's Rule which uses parabolas. I feel these methods are best used with computer software. I have also discovered with modern computers, you can simply use more rectangles, and it will give approximations as good as any method. This is due to the fact that since modern computers are now so powerful, you can write a simple program using millions rectangles that will calculate an answer nearly instantaneously.

EXAMPE:

Take a look at the graph below to see a better representation of approximating an area using rectangles.



EXAMPLE:

To become a little more familiar with the summation notation, let us write the following sums using summation notation:

$$1 + 2 + 3 + 4 + 5 = \sum_{i=1}^{5} i$$

$$f(1) + f(2) + f(3) = \sum_{i=1}^{3} f(i)$$

EXERCISES:

Use summation notation to rewrite the following sums:

1) $2 + 4 + 6 + 8 + 10 + 12$

2) $f(x_0) + f(x_1) + f(x_2) + f(x_3)$

3) $1 + 3 + 5 + 9 + 11 + 13 + 15 + 17$

Rewrite the following sums without summation notation (i.e. write out all the terms):

4) $\sum_{i=1}^{5} i^2$

5) $\sum_{i=1}^{3} f(x_i)$

6) $\sum_{i=1}^{4}(1 + i)$

Use a) left endpoints and b) right endpoints to approximate the area under the curve (and above the x-axis) of the following functions over the given interval using 4 rectangles:

7) $f(x) = \frac{1}{x},\ [1,5]$

8) $f(x) = x^2,\ [0,4]$

9) $f(x) = \sin x,\ \ [0, \pi]$

CHAPTER 4
SECTION 2
THE DEFINITE INTEGRAL

In Section 1, we approximated the area under a curve, and above the x-axis, with the Riemann Sum, $\sum_{i=1}^{n} f(x_i)\Delta x$. We found that it may or may not be a very good approximation for the area, depending on the number of rectangles used. What if we had an infinite amount of rectangles? It is fairly easy to imagine that this would give us an accurate representation of the area. We now have an infinite number of rectangles, which implies all the widths, i.e. Δx, are going to zero.

We now define the area under the curve and above the x-axis, over an interval I to be as such:

Let f be continuous on a closed interval $I = [a, b]$, then the area bounded by $f(x) \geq 0$, the x-axis, and the lines $x = a,\ x = b$ is as follows:

$\boldsymbol{A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}$, provided the limit exists.

DEFINITION OF A DEFINITE INTEGRAL:

If f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal width $\frac{b-a}{n}$, and $x_1, x_2, x_3, \dots, x_n$ are the sample points of the subintervals, then the definite integral of f from a to b, is $\boldsymbol{\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}$, provided the limit exists, (and gives the same value for all choices of sample points). If these facts hold true, we say f is integrable on $[a, b]$.

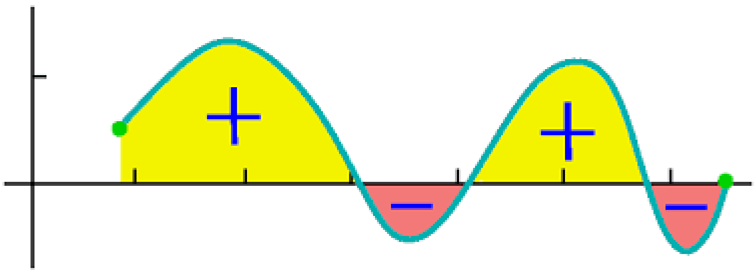
Let's look at the notation: $\boldsymbol{\int_a^b f(x)dx}$: This reads: **"The integral from $\boldsymbol{a}$ to $\boldsymbol{b}$ of $\boldsymbol{f(x)dx}$**". Note that a, b are the endpoints of the interval. f is the height, and dx is the width. Also note that dx is very small, in fact, it is infinitesimally small. (Note, the integral symbol $\int$ was invented by Leibnitz, whom we have previously discussed).

We observe that a and b are called the "limits of integration", with a being the lower limit, and b being the upper limit.

NOTE ON FUNCTIONS OVER AN INTERVAL THAT OCCUR BOTH ABOVE AND BELOW THE X-AXIS:

So far we have understood area to be $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, which means the area below $f(x)$ and above the x-axis over the interval $[a, b]$. But what happens if a function has both positive and negative values over the interval? We sum the rectangles above the x-axis, and the negatives of the sums below the x-axis, to obtain what we refer to as "net area". E.g., $\int_a^b f(x)dx = A_2 - A_1$, where A_2 represents the area above the x-axis, and A_1 represents the area below the x-axis. We note that, we do NOT have a negative area. The integral represents the area above the x-axis minus the area below the

x-axis. It does not mean the area itself is negative. It means the NET area can be positive, negative, or zero; depending on how much is above, and how is below the x-axis.



FORMULAS FOR EVALUATING INTEGRALS:

1) We begin with $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

2) To evaluate number 1), we will need $\Delta x = \frac{b-a}{n}$,

3) We also need $x_i = a + i\Delta x$

We also need formulas for powers of positive integers:

4) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

5) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

6) $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

And, we will need some reminders:

7) $\sum_{i=1}^{n} c = nc$, where c is a constant

8) $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

9) $\sum_{i=1}^{n} a_i \pm b_i = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

EXAMPLE:

Let us find $\int_0^4 (x^2+1)dx$, i.e. find the area for the problem we approximated in Section 1.

$\int_0^4 (x^2+1)dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i^2+1)\Delta x.$

Now, we have $\Delta x = \frac{4-0}{n} = \frac{4}{n}$.

We have $x_i = a + i\Delta x = 0 + i \cdot \frac{4}{n} = \frac{4i}{n}$.

So $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^2 + 1)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{4i}{n}\right)^2 + 1\right] \cdot \frac{4}{n}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{16i^2}{n^2} + 1\right] \cdot \frac{4}{n} =$$

$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{4}{n} + \frac{64i^2}{n^3}\right] =$ (By sum formula 9) and limit law 1)): =

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} + \lim_{n\to\infty} \sum_{i=1}^{n} \frac{64i^2}{n^3} =$ (By sum formulas 7) and 8)): =

$$\lim_{n\to\infty} \frac{4}{n} \cdot n + \lim_{n\to\infty} \frac{64}{n^3} \sum_{i=1}^{n} i^2$$

Now, we must use our summation formula for i^2:

We substitute $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ into the above expression:

$$4 + \lim_{n\to\infty} \frac{64}{n^3} \sum_{i=1}^{n} i^2 =$$

$$4 + \lim_{n\to\infty} \frac{64}{n^3} \cdot \left(\frac{n(n+1)(2n+1)}{6}\right) =$$

$$4 + \frac{128}{6} = 4 + \frac{64}{3} = \frac{76}{3}.$$

Note: Recall from the previous section, that when we approximated this area using 4 rectangles, we got 18 for the underestimate, and 34 for the overestimate. $\frac{76}{3} = 25\frac{1}{3}$ which falls really close to the midpoint between these values. If we had used the midpoint rule, we would have gotten a more reasonable answer only using 4 rectangles.

PROPERTIES OF THE DEFINITE INTEGRAL:

1) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2) $\int_a^a f(x)\,dx = 0$

3) $\int_a^b c\,dx = c\,(b-a)$

4) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

5) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

6) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$ (provided $a < c < b$)

7) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

8) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

EXAMPLE:

Let's use the properties of integrals to evaluate $\int_0^4 (4x^2 + 7)\,dx$:

First, we will use properties 3) and 4), to rewrite $\int_0^4 (4x^2 + 7)\,dx$ as $4\int_0^4 (x^2 + 1)\,dx + \int_0^4 3\,dx$.

$\int_0^4 3\,dx = 3 \cdot 4 = 12$ by property 3). (We also observe why property 3 works: It is the area of a rectangle).

You might wonder why we broke up the integral in this exact manner. We did it, because we calculated $\int_0^4 (x^2 + 1)\,dx = \frac{76}{3}$ in the previous example; and we can use that result here.

Therefore, $\int_0^4 (4x^2 + 7)\,dx = 12 + 4 \cdot \frac{76}{3} = \frac{340}{3}$.

EXAMPLE:

If $\int_1^2 f(x)\,dx = 3$, and $2\int_2^7 f(x)\,dx = 2$, what is $\int_1^7 f(x)\,dx$? Using properties 4) and 5): $\int_1^7 f(x)\,dx = 3 + \frac{2}{2} = 4$.

EXERCISES:

Rewrite the following limits as definite integrals:

1) $\lim_{n\to\infty} \sum_{i=1}^{n} (3x_i^5 + 11x_i^3)\Delta x, \ \ [1,2]$

2) $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(x_i^2 - 3)}\Delta x, \ \ [3,5]$

3) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(e^{2x_i} + x^{\frac{1}{3}}\right)\Delta x, \ \ [0,1]$

4) $\lim_{n\to\infty} \sum_{i=1}^{n} (\sin x_i^2 + 1)\Delta x, \ \ [0,\pi]$

Use the definition of the integral to evaluate the following: (Hint: this is the limit of the Riemann Sum):

5) $\int_0^2 (x-1)\,dx$

6) $\int_1^2 (2x+1)\,dx$

7) $\int_0^2 (x^2+x)\,dx$

8) $\int_1^2 (x^2-1)\,dx$

9) $\int_1^2 (x^2-x+1)\,dx$

Evaluate the following integrals by combining the properties for definite integrals along with your knowledge of areas:

10) $\int_1^2 (x-1)\,dx$ (Hint: You should get the same answer you did in number 5) with less work!)

11) $\int_1^2 (2x+1)\,dx$ (Hint: You should get the same answer you did in number 6) with less work!)

12) $\int_{-1}^{1} \left(4 + \sqrt{1-x^2}\right)dx$

Use properties of definite integrals to evaluate the integrals:

13) If $\int_0^2 f(x)\,dx = 5$, and $3\int_2^4 f(x)\,dx = 9$, what is $\int_0^4 f(x)\,dx$?

14) If $\int_0^7 f(x)\,dx = 8$, and $3\int_0^4 f(x)\,dx = 3$, what is $\int_4^7 f(x)\,dx$?

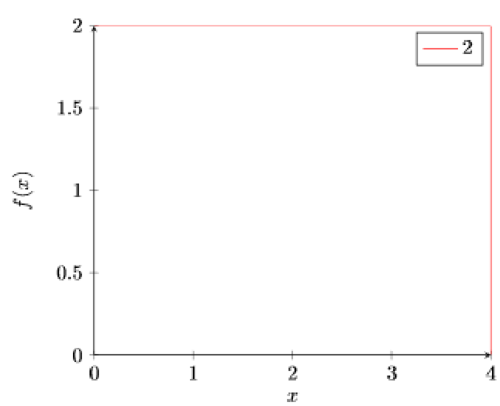
CHAPTER 4
SECTION 3
THE FUNDAMENTAL THEOREM OF CALCULUS

So there are two Fundamental Theorems of Calculus. They are called the First Fundamental Theorem of Calculus, and the Second Fundamental Theorem of Calculus. Some texts label them the opposite of other texts.

I would like to start with the one I feel follows best from what we just learned. Since I will present this first, I will call it The First Fundamental Theorem of Calculus, by default.

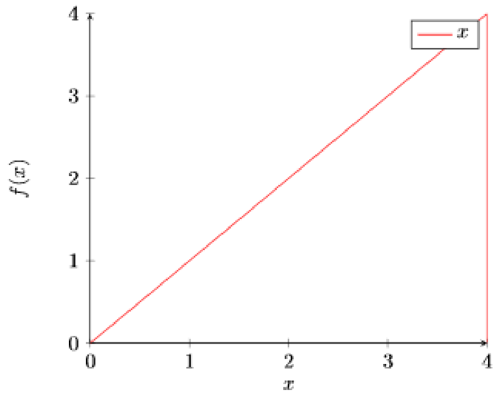
Let us review what we have learned thus far in this chapter: We first learned that we could approximate an area with a Riemann Sum: $\sum_{i=1}^{n} f(x_i)\Delta x$ in Section 1. We then learned that we could find the actual area by taking the limit: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, in Section 2. We also learned in Section 2, that the definition of the definite integral was this limit: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$. Next, we calculated the integral by using the limit definition, and used some sum formulas to calculate the limit. We did a simple example, and some simple exercises to perform this task. We noticed that, even with the simplest exercises, this task became quite cumbersome and tedious. Imagine if we had to do much more complicated integrals? E.g., even a slightly higher polynomial would become quite tedious very quickly.

Next, to warm up to the idea of the First Fundamental Theorem of Calculus, let's view some examples from the lens of area:



Let's call the width x, and the height m: We note the area is $A = mx$

Let us now graph the new function: $f = mx$:



Let's calculate this area: $A = \frac{1}{2}\, l \cdot w = \frac{1}{2}\, mx \cdot x = \frac{mx^2}{2}$

Perhaps you notice a pattern. The area of the rectangle $f = m$ was mx.
The area of "that area function" was $\frac{mx^2}{2}$. Do you notice that each area was the antiderivative of the function? Can you guess the area of the function: $f = \frac{mx^2}{2}$? Hopefully you guessed it would be $\frac{mx^3}{6}$. That would be correct. Recall we learned about antiderivatives in Chapter 3.7.

This awakens us to the idea behind the First Fundamental Theorem of Calculus, which we will formalize and prove below:

<u>THE FIRST FUNDAMENTAL THEOREM OF CALCULUS:</u>

If f is continuous on the closed interval $[a, b]$, then $\int_a^b f(x)\, dx = F(b) - F(a)$, where F is any antiderivative of $f \rightarrow \quad F' = f$.

(We recall $\int_a^b f(x)\, dx$ is the area under the curve over $[a, b]$, and F is the antiderivative of f). (Also note that we evaluate F at the endpoints to find the area).

PROOF:

We start by writing $F(b) - F(a)$ in a different form. Let us partition the interval $[a, b]$ into:

$a = x_0 < x_1, x_2 < \cdots < x_{n-1} < x_n = b.$

Next, we can write $F(b) - F(a)$ as pairwise addition and subtraction:

$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdots - F(x_1) + F(x_1) - F(x_0) =$

$\sum_{i=1}^{n}[(F(x_i) - F(x_{i-1})]$.

Now we apply the Mean Value Theorem to show there exists a number c such that

$F'(c_i) = \frac{F(x_i)-F(x_{i-1})}{x_i-x_{i-1}}$, where c_i is in the ith subinterval (recall i can be any of $i = 0,1,2, \dots, n$, i.e. It is arbitrary).

We know we can write $F'(c_i)$ as $f(c_i)$, and we let $\Delta x = x_i - x_{i-1}$. (Which simply means Δx is the change in x in one of these subintervals).

Therefore, $F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x$.

This means you can keep applying the Mean Value Theorem to find a collection of c_i's So that $F(b) - F(a)$ is a Riemann sum of f on $[a, b]$ for any partition.

Observe what we are doing: We are finding the area of a rectangle and adding all the areas together. Each rectangle describes an approximation of the curve of the section it is placed under. We also note that the width of the rectangles do not have to be the same. Recall, in the previous section that we discussed the more rectangles, the better the approximation. When we had an infinite number of rectangles; i.e., we took the limit as $n \to \infty$, we got the integral. As we approach ∞ for the number of rectangles, the limit of $\|\Delta x_i\| \to 0$. What does that mean? It means the widths of each rectangle approaches zero. This makes sense, as the more rectangles we have, the smaller each width will be.

So taking the limit as $n \to \infty$ also implies the limit as $\|\Delta x_i\| \to 0$, and taking the limit of both sides gives us:

$$\lim_{\|\Delta x_i\| \to 0} F(b) - F(a) = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x \to \quad F(b) - F(a) = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\, dx.$$

STEPS FOR APPLYING THE FUNDAMENTAL THEOREM OF CALCULUS:

1) $\int_a^b f(x)\, dx = F(x)]_a^b$. We took the general antiderivative of $f(x)$. In doing so, the $\int$ symbol disappeared, as did the dx. We also placed a bar after $F(x)$ with the limits of integration in the same positions.

2) Next, we calculate $F(b) - F(a)$. Note that we substitute the top number into F and then subtract off $F(a)$, (the value at the bottom). We get a numerical answer.

3) You might ask why we don't have a "$+ C$", which we discovered we needed in Section 3.7. Observe since we now have subtraction $(F(x) + C)]_a^b =$

   $F(b) + C - (F(a) + C) = F(b) - F(a)$.

EXAMPLE:

1) Use the Fundamental Theorem of Calculus to evaluate the following integrals:

   a) $\int_0^1 2x\,dx = \frac{2x^2}{2}\Big|_0^1 = x^2\big]_0^1 = 1^2 - 0^2 = 1.$ So let's go over these steps:

   First we took the antiderivative: x^2. Notice the integral symbol $\int$ disappeared as did the dx.

   Next, we observe that after the antiderivative, there is this bar with the limits of integration, $\big]_0^1$.

   Then we performed the $F(b) - F(a)$ part, substituting the top number, and then subtracting the bottom number substituted.

   b) $\int_0^4 (x^2+1)\,dx$: Note that this is the same problem we have done. We approximated its area in Section 1, and we found the area using the limit definition in Section 2. We shall get the same results we got in Section 2 using the Fundamental Theorem of Calculus, and we will see how much easier it is than using the limit definition.

   $\int_0^4 (x^2+1)\,dx = \left(\frac{x^3}{3}+x\right)\Big|_0^4 = \left(\frac{4^3}{3}+4\right) - \left(\frac{0^3}{3}+0\right) = \frac{64}{3}+4 = \frac{76}{3}.$ Notice it is the same answer from Section 2. Also note how much easier and quicker it was. We love the Fundamental Theorem of Calculus!

   c) $\int_0^{\frac{\pi}{2}} \cos x\,dx = \sin x\big]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1.$

   d) $\int_0^1 (e^x+1)\,dx = (e^x+x)\big]_0^1 = (e^1+1) - (e^0+0) = e$

2) Find the area of the following functions over the given interval:

   a) $y = \sin x + x,\ \ [0,\pi]$: We set up the integral $\int_0^\pi (\sin x + x)\,dx$. Next, we evaluate the integral: $\int_0^\pi (\sin x + x)\,dx = \left(-\cos x + \frac{x^2}{2}\right)\Big|_0^\pi = \left(-\cos\pi + \frac{\pi^2}{2}\right) - \left(-\cos 0 + \frac{0^2}{2}\right) = 1 + \frac{\pi^2}{2} + 1 = \frac{\pi^2}{2} + 2.$

   b) $y = x^2,\ \ y = 0,\ \ x = 1.$ First we must find the bound (limits of integration) in x: when $y = 0, x = 0$. So we integrate over $[0,1]$.

   Next, we set up the integral $\int_0^1 x^2\,dx$.

Now we evaluate the integral $\int_0^1 x^2\, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$.

THE SECOND FUNDAMENTAL THEOREM OF CALCULUS: (Sometimes referred to as the First Fundamental Theorem of Calculus):

For this Theorem, we begin by looking at what we have done thus far: We have evaluated integrals where the limits of integration were constants, i.e. of b being the upper limit. We now want to look at x (a variable) being the upper limit. We don't want to confuse ourselves by using x as both the upper limit and also as the variable of integration, so we write: $F(x) = \int_a^x f(t)\, dt$, where f is a continuous function on $[a, b]$, and $a \le x \le b$. If f is positive, we view $F(x)$ as the area under the curve of f from a to x, with x being able to vary from a to b rather than having to be held constant.

THE THEOREM STATES:

If f is continuous on $[a, b]$, then F is defined by $F(x) = \int_a^x f(t)\, dt, \quad a \le x \le b$, is continuous on $[a, b]$ and differentiable on (a, b), and $F'(x) = f(x)$.

PROOF:

We begin with the definition: $F(x) = \int_a^x f(t)\, dt$

If x and $x + h$ are in (a, b), then we take the derivative of $F(x)$

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} =$$

$$\lim_{h \to 0} \frac{\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt}{h}$$

$$\lim_{h \to 0} \frac{1}{h} \cdot \left[\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt \right] =$$

$$\lim_{h \to 0} \frac{1}{h} \cdot \left[\int_a^x f(t)\, dt + \int_x^{x+h} f(t)\, dt - \int_a^x f(t)\, dt \right] =$$ (From Property 6) in Section 2)

$$\lim_{h \to 0} \frac{1}{h} \cdot \int_x^{x+h} f(t)\, dt$$

We will assume $h > 0$. Since f is continuous on $[a, b]$, and both x and $x + h$ are in $[a, b]$, then f is continuous on $[x, x + h]$.

The Extreme Value Theorem guarantees there is an absolute maximum and an absolute minimum in $[x, x+h]$. We let u be the value of x in which the absolute minimum occurs, and v be the value of x in which the absolute maximum occurs. We let $f(u) = m$, the absolute minimum, and $f(v) = M$, the absolute maximum.

There is another comparison property of Integrals that states:

If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a)$.

We apply this property to get:

$$mh \leq \int_x^{x+h} f(t)\, dt \leq Mh \rightarrow$$

$$f(u)h \leq \int_x^{x+h} f(t)\, dt \leq f(v)h \rightarrow$$

$f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)\, dt \leq f(v)$ (Since we assumed $h > 0.$)

Next recall: $\frac{F(x+h)-F(x)}{h} = \int_x^{x+h} f(t)\, dt.$

Therefore: $f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)\, dt \leq f(v) =$

$$f(u) \leq \frac{F(x+h)-F(x)}{h} \leq f(v)$$

Next we take $\lim_{h\to 0}\left[f(u) \leq \frac{F(x+h)-F(x)}{h} \leq f(v)\right]$

Since u and v are both in $[x, x+h]$, then

$\lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x)$ and

$\lim_{h\to 0} f(v) = \lim_{v\to x} f(v) = f(x)$

Because f is continuous at x. (We also note that $h \to 0$ implies that the interval $[x, x+h]$ is so small that all values in this interval are all approaching each other, hence $\lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x)$ and $\lim_{h\to 0} f(v) = \lim_{v\to x} f(v) = f(x)$).

Hence, $\lim_{h\to 0}\left[f(u) \leq \frac{F(x+h)-F(x)}{h} \leq f(v)\right] \rightarrow$

$$f(x) \leq \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \leq f(x) \rightarrow$$

$\lim_{h \to 0} \frac{F(x+h)-F(x)}{h} = f(x)$ **By the Squeeze Theorem**.

Now we observe that $\lim_{h \to 0} \frac{F(x+h)-F(x)}{h} = F'(x)$,

So, $F'(x) = f(x)$ where $F(x) = \int_a^x f(t)\,dt$.

We note that the integral of the derivative is the original function f. This also makes intuitive sense to us. (Also note, we can do another proof for $h < 0$, which is omitted here).

EXAMPLE:

Find $F'(x)$ if $F(x) = \int_1^x (t^2 + 1)\,dt$: Since $f(t)$ is continuous, we get $F'(x) = x^2 + 1$. (All we did was replace x for t into f).

EXAMPLE:

Find $F'(x)$ if $F(x) = \int_1^{x^2} \sin t \;\, dt$: We notice something different about this problem right away. Hopefully you observed that the upper limit of integration is x^2 instead of x. The theorem applies to x as the upper limit. So what do we do? We will do a substitution: Let $u = x^2$. We next take the derivative: $\frac{du}{dx} = 2x$.

So we get $\frac{d}{dx}\left(\int_1^{x^2} \sin t \;\, dt\right) = \frac{d}{du}\left(\int_1^{u} \sin t \;\, dt\right) \cdot \frac{du}{dx}$ $\text{(Chain Rule)} = \sin u \cdot \frac{du}{dx} = (\sin u) \cdot 2x =$ $\mathbf{2x \sin x^2}$.

SUMMARY:

THE FUNDAMENTAL THEOREM OF CALCULUS:

If f is continuous on the closed interval $[a, b]$:

- Then $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of $f \rightarrow \quad F' = f$.
- And, $F(x) = \int_a^x f(t)\,dt, \quad a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b), and $F'(x) = f(x)$.

EXERCISES:

Evaluate the following integrals using the Fundamental Theorem of Calculus:

1) $\int_0^1 (x^2 - 4x + 7)\, dx$

2) $\int_0^1 (2x - 5)\, dx$

3) $\int_1^2 (4x^3 - 3x^2 + x - 1)\, dx$

4) $\int_0^2 (x^4 - 2x + 3)\, dx$

5) $\int_{-1}^1 x^{\frac{1}{3}}\, dx$

6) $\int_1^2 \left(\sqrt{x} - \frac{2}{x} + \frac{1}{x^2}\right)\, dx$

7) $\int_0^2 (x - 2)(x + 3)\, dx$

8) $\int_0^1 \left(\frac{1}{2}e^x - x^{\frac{5}{2}} + 4x\right)\, dx$

9) $\int_0^\pi (\sin x + \sec^2 x + e^x)\, dx$

10) $\int_1^4 \left(\sqrt{2}\, x - \frac{1}{x^{\frac{2}{3}}}\right)\, dx$

11) $\int_1^2 \frac{x^2 - 2x + x^{-1}}{x}\, dx$

12) $\int_1^3 \frac{x^2 - 2x}{\sqrt{x}}\, dx$

13) $\int_0^1 3^x\, dx$

14) $\int_0^3 x^2(2x - 4)\, dx$

15) $\int_0^{\frac{\pi}{4}} (\cos x + \sec x \tan x)\, dx$

16) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2 \csc^2 x\, dx$

17) $\int_0^1 \left(\frac{1}{1+x^2} - 2x\right)\, dx$

18) $\int_0^{\frac{\pi}{2}} \left(\frac{1}{\sqrt{1-x^2}} + e^x\right)\, dx$

19) $\int_0^1 (x^2 + 2^x)\, dx$

20) $\int_0^1 \sqrt{2}\pi\, dt$

Find the area of the following functions over the given interval: (Hint: for some problems, you may have to find the interval from the information given).

21) $y = 2x^3,\ [0,1]$

22) $y = e^x,\ [0,2]$

23) $y = \cos x + x^2,\ \left[0, \frac{\pi}{4}\right]$

24) $y = \sec^2 x,\ \left[0, \frac{\pi}{3}\right]$

25) $y = x^3,\ y = 0,\ x = 2$

26) $y = x - x^2,\ y = 0$

Find the derivative of the function:

27) $F(x) = \int_2^x \sqrt{3t^2 + 4t}\, dt$

28) $F(x) = \int_{100}^x (\sin^2 t + \cos^3 t)\, dt$

29) $F(x) = \int_\pi^x \frac{\ln t}{\sqrt{t}}\, dt$

30) $F(x) = \int_1^{x^2} \cos t\, dt$

31) $F(x) = \int_2^{\sqrt{x}} (2t - e^t)\, dt$

32) $F(x) = \int_x^2 (t^2 + \sin^{-1} t)\, dt$ (Hint: Rewrite the integral using one of the properties of definite integrals).

33) $F(x) = \int_{x^2}^x \sin t\, dt$ (Hint: Rewrite using two of the properties of definite integrals).

CHAPTER 4
SECTION 4
INDEFINITE INTEGRALS AND THE NET CHANGE THEOREM

The Fundamental Theorem of Calculus showed connections between antiderivatives and definite integrals. Because of this, the natural notation for the general antiderivative is an integral that we call an Indefinite Integral. This means it does not have limits of integration. It is not a number (nor an area), but a general antiderivative in which we will have a $+C$.

$$\int f(x)\, dx = F(x) \rightarrow \quad F'(x) = f(x).$$

The Indefinite Integral represents a "family of functions", with different values for C. So the Definite Integral represents a value, or area; and the Indefinite Integral represents a function, or family of functions.

We now make a table of Indefinite Integrals:

$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \quad n \neq 1$
$\int e^x\, dx = e^x + C$
$\int a^x\, dx = \frac{a^x}{\ln a} + C$
$\int \frac{1}{x}\, dx = \ln\lvert x\rvert + C$
$\int \cos x\; dx = \sin x + C$
$\int \sin x\, dx = -\cos x + C$
$\int \sec^2 x\, dx = \tan x + C$
$\int \sec x \tan x\; dx = \sec x + C$
$\int \frac{1}{\sqrt{1-x^2}}\, dx = sin^{-1} x + C$
$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$
$\int \cosh x\; dx = \sinh x + C$
$\int \sinh x\; dx = \cosh x + C$
$\int cf(x)\, dx = c\int f(x)\, dx$
$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x) dx$

EXAMPLE:

Evaluate the following Indefinite Integrals:

1) $\int (x^2 - 2x)\, dx = \frac{x^3}{3} - \frac{2x^2}{2} + C = \frac{x^3}{3} - x^2 + C$
2) $\int \sin x\, dx = -\cos x + C$

3) $\int \left(x^{\frac{1}{3}} - \frac{1}{x} + e^x\right)\, dx = \frac{3}{4}x^{\frac{4}{3}} - \ln|x| + e^x + C$

4) $\int \left(\sec x \tan x + 3^x - \frac{1}{1+x^2}\right) dx = \sec x + \frac{3^x}{\ln 3} - \arctan x + C$

NET CHANGE THEOREM:

The Integral of a rate of change is the net change:

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Application of the Net Change Theorem:

Recall, that if an object moves along a straight line by a function $s(t) > 0$, then $v(t) = s'(t)$.
So $\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1)$ is the net change, or displacement from time $= t_1$ to time $= t_2$.

To calculate the total distance traveled, we have to consider $v(t) \leq 0$ and $v(t) \geq 0$. Recall velocity can be positive or negative depending on if we are moving forward or backward. Displacement can also be positive or negative (positive when moving right, and negative when moving left). Distance, however, is never negative, (as it is a length). So distance is $\int_{t_1}^{t_2} |v(t)|\, dt$. (We also refer to $|v(t)|$ as speed).

Velocity can be found from acceleration in the same manner.

There are many other applications of the Net Change Theorem, which can be explored by the student.

EXAMPLE:

A particle moves in a straight line such that its velocity (as a function of time $= t$) is $v(t) = t^2 - 5t - 6$. (In meters/second).

a) Find the displacement for $0 \leq t \leq 3$
b) Find the total distance traveled during this time.

a) Displacement is $\int_0^3 (t^2 - 1)\, dt = \left(\frac{t^3}{3} - t\right)\Big|_0^3 = 9 - 3 = 6$. This means the particle ends up 6 meters to the right of where it started (after $3\ s$), since the displacement is positive.

b) The total distance traveled: $v(t) = t^2 - 1 = (t-1)(t+1)$, implies $v(t) \leq 0$ on $[0,1]$ and $v(t) \geq 0$ on $[1,3]$.

So distance $= \int_0^1 -(t^2 - 1)\, dt + \int_1^3 (t^2 - 1)\, dt = \left(-\frac{t^3}{3} + t\right)\Big|_0^1 + \left(\frac{t^3}{3} - t\right)\Big|_1^3 = -\frac{1}{3} + 1 + 9 - 3 - \frac{1}{3} + 1 = \frac{22}{3}$ meters traveled.

EXERCISES:

Find the general indefinite integral:

1) $\int (3x^2 - 4x + 8)\, dx$

2) $\int (x^3 - 2x^2 + 4x - 9)\, dx$

3) $\int (5x^4 - 2x^2 + 12x)\, dx$

4) $\int (2x^{-3} + e^x + 7)\, dx$

5) $\int (x - 1)(x + 3)\, dx$

6) $\int (x^2 + 2)(x - 5)\, dx$

7) $\int \left(2\sqrt{x} - \frac{3}{x} + x^{-2}\right) dx$

8) $\int (2 \sin x - 4 \cos x)\, dx$

9) $\int (2e^x + \sec^2 x)\, dx$

10) $\int \left(\frac{5}{x} - \frac{2}{x^2} + \frac{1}{x^3} - \frac{x}{3}\right) dx$

11) $\int \left(\sec x \tan x - \frac{1}{1+x^2} + x^{\frac{1}{3}}\right) dx$

12) $\int \left(\sqrt[3]{x^2} - x^{-5} - \sqrt{2}\right) dx$

13) $\int \left(\frac{x^{\frac{1}{2}} - 2x + 7}{x}\right) dx$

14) $\int \left(\frac{x - 5 - x^{\frac{1}{3}}}{x^2}\right) dx$

15) $\int \left(\frac{1}{\sqrt{1-x^2}} + \sec^2 x + \frac{2}{x}\right) dx$

16) $\int (7^x + 7e^x + x^7)\, dx$

17) A particle moves in a straight line such that its velocity (as a function of time $= t$ is $v(t) = t^2 - 5t - 6$. (In meters/second).

a) Find the displacement for $0 \leq t \leq 10$

b) Find the total distance traveled during this time.

18) A particle moves in a straight line such that its velocity (as a function of time $= t$ is $v(t) = t^2 - 4$. (In meters/second).

a) Find the displacement for $0 \leq t \leq 5$
b) Find the total distance traveled during this time.

CHAPTER 4
SECTION 5
THE SUBSITUTION RULE FOR INTEGRALS

The Substitution Rule for Integrals is often referred to as "U-Substitution". What is this used for? Recall the Chain Rule for Derivatives. U-Substitution is basically a backwards Chain Rule for Antiderivatives.

EXAMPLE:

$f(x) = (x^2 + 2)^{10} \rightarrow \quad f'(x) = 10(x^2 + 2)^9(2x)$. Hopefully, this looks familiar. We used the Chain Rule to find the derivative.

For integrals using "U-Substitution", we take something that looks like the derivative function on the right, and make it look like the function on the left.

Let's simplify the $f'(x)$ function just slightly: $\int (x^2 + 2)^9(2x)\, dx$.

First, we choose our u: We choose $u = x^2 + 2$. Why did we do that? Because $\frac{du}{dx} = 2x$. We observe this to be the other function. (We will see in a minute, that they will cancel).

From $u = x^2 + 2$, we find the differential: $du = 2x\, dx$. Next we solve for dx. $dx = \frac{du}{2x}$.

Next, we substitute everything we have: $\int [u^9(2x)] \frac{du}{2x} = \int u^9\, du = \frac{u^{10}}{10} + C$.

Lastly, we back substitute in terms of x: $u = x^2 + 2 \rightarrow \frac{u^{10}}{10} + C = \frac{(x^2+2)^{10}}{10} + C$.

(Note if you take the derivative, you will get: $(x^2 + 2)^9(2x)$).

THE SUBSITUTION RULES FOR INTEGRALS:

If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then $\int f\big(g(x)\big)g'(x)\, dx = \int f(u)\, du$.

STEPS FOR USING U-SUBSITUTION:

1) Choose your u. Choose it to be the function whose derivative is the other function. (Note: This may be immediately obvious, or it could take some trial and error: a bit like "guess and check" for factoring.) (Also note that the derivative might have a different constant multiple, and that's okay).

2) Take the differential: i.e. $du = g'(x)\ dx$

3) Substitute u, and du into the integral.

4) Cancel everything that will cancel. (You should now only have one variable: u. All the x-terms should cancel. If they don't, you probably chose the wrong function to be your u. Go back and choose something else). (In some instances, you did not choose incorrectly, and other methods need to be additionally employed).

5) Integrate with respect to u.

6) Back substitute in terms of x.

EXAMPLE:

Find the following indefinite integrals using u-substitution:

1) $\int (x^3 - 5)^5 x^2\ dx$. We choose $u = x^3 - 5$. Do you see why? Its derivative is $3x^2$, which is a constant multiple times x^2, which is our $g'(x)$.

   Next, we take the differential: $du = 3x^2\ dx$.

   Solve for dx. $dx = \frac{du}{3x^2}$.

   Now, we substitute: $\int u^5 \cdot x^2 \cdot \frac{du}{3x^2} = \frac{1}{3}\int u^5 du$. (Notice that the x^2 canceled, and we pulled the $\frac{1}{3}$ out in front of the integral from property 13 from Section 4.

   Now, we will integrate: $\frac{1}{3}\int u^5 du = \frac{u^6}{18} + C$.

   Lastly, we back substitute: $\frac{(x^3-5)^6}{18} + C$.

2) $\int 2x \cdot \cos x^2\ dx$. Choose $u = x^2$, because $\frac{d}{dx}(x^2) = 2x$, and it will cancel.

   Next, $du = 2x\ dx \rightarrow \quad dx = \frac{du}{2x}$

   Substitute: $\int 2x \cos u \frac{du}{2x} = \int \cos u\ \ du = \sin u + C = \sin x^2 + C$.

3) $\int e^{2x} dx$. We choose $u = 2x$. We observe that this works because $\frac{d}{dx}(2x) = 2$, which is a constant.

$du = 2\,dx \rightarrow \quad dx = \dfrac{du}{2}$

Substitute: $\int e^u \frac{du}{2} = \frac{1}{2}\int e^u\, du = \frac{1}{2}\, e^u + C = \frac{1}{2} e^{2x} + C.$

Let us note that for $\int \boldsymbol{a \cdot e^{bx} dx = \frac{a}{b}\, e^{bx} + C}$. We can prove this using U-substitution:

Let $u = bx \rightarrow \quad du = b\, dx \rightarrow \quad dx = \frac{du}{b}$

$\int a \cdot e^{bx} dx = a\int e^u \frac{du}{b} = \frac{a}{b}\int e^u\, du = \frac{a}{b}\, e^u + C = \frac{a}{b}\, e^{bx} + C.$ (We also note that $\frac{d}{dx}\left(\frac{a}{b}\, e^{bx}\right) = a\, e^{bx}$.)

We can now apply this rule instead of using U-substitution each time we have an integral of this form.

4) $\int \frac{2x}{\sqrt{x^2-2}}\, dx.$ Choose $u = x^2 - 2$.

$du = 2x\, dx \rightarrow \quad dx = \dfrac{du}{2x}.$

Substitute: $\int \frac{2x}{\sqrt{x^2-2}}\, dx = \int \frac{2x}{u^{\frac{1}{2}}}\frac{du}{2x} = \int u^{-\frac{1}{2}}\, du = 2\, u^{\frac{1}{2}} + C = 2\sqrt{x^2-2} + C$

5) $\int \sqrt{x+1}\, x^2\, dx.$ This one involves a little trick: Choose $u = x + 1 \rightarrow \quad du = dx.$

Okay, so far so good, right? But what about the x^2? It's not going to cancel.

Recall, that we need to have a single variable to integrate an integral (At least for a single integral. Double and triple integrals will use multi-variables, which we discover later in a Calculus 3 course).

The trick: If $u = x + 1$, then $x = u - 1$.

So, $\int \sqrt{x+1}\, x^2\, dx = \int u^{\frac{1}{2}}(u-1)^2\, du = \int u^{\frac{1}{2}}(u^2 - 2u + 1)\, du = \int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du.$

Now, we have a form that we can integrate:

$$\int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du = \frac{2}{7}u^{\frac{7}{2}} - 2 \cdot \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C$$
$$= \frac{2}{7}(x+1)^{\frac{7}{2}} - \frac{4}{5}(x+1)^{\frac{5}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + C.$$

6) $\int \tan x \, dx$. First we rewrite $\tan x = \frac{\sin x}{\cos x}$

So $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$.

Choose $u = \cos x$ (Note: It is common that u will be the denominator in a rational function).

Then, $du = -\sin x \, dx \rightarrow \quad dx = -\frac{du}{\sin x}$

Substitute: $\int \frac{\sin x}{\cos x} dx = -\int \frac{\sin x}{u} \frac{du}{\sin x} = -\int \frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C =$

$\ln(|\cos x|)^{-1} + C = \ln\left(\frac{1}{|\cos x|}\right) + C = \ln|\sec x| + C.$

SUBSTITUTION FOR DEFINITE INTEGRALS:

Let's return to the first example we had: $\int (x^2+2)^9(2x)\, dx$.

Let us now revise it by putting in some limits of integration:

$$\int_0^1 (x^2+2)^9(2x)\, dx.$$

There are two ways to evaluate this integral:

The first is to evaluate it as an indefinite integral. We did this above by applying U-substitution:

We got: $\int (x^2+2)^9(2x)\, dx = \frac{(x^2+2)^{10}}{10} + C$. We can now apply the Fundamental Theorem of Calculus to get: $\int_0^1 (x^2+2)^9(2x)\, dx = \frac{(x^2+2)^{10}}{10}\Big|_0^1 = \frac{3^{10}}{10} - \frac{2^{10}}{10} = \frac{59{,}049}{10} - \frac{1024}{10} = \frac{58{,}025}{10} = \frac{11{,}605}{2}$.

There is another, easier (in my opinion), way to evaluate this integral. I believe it is easier, because it eliminates a couple of steps.

The way to do this, is to change the limits of integration from x to u, when we do the original substitution:

We chose $u = x^2 + 2$. This implies when $x = 0 \rightarrow u = 2$, and when $x = 1 \rightarrow u = 3$.

So $\int_0^1 (x^2+2)^9(2x)\, dx = \int_2^3 u^9 du = \frac{u^{10}}{10}\Big|_2^3 = \frac{3^{10}}{10} - \frac{2^{10}}{10} = \frac{11{,}605}{2}$.

We note that this method was a bit shorter and simpler.

THE SUBSTITUTION RULE FOR DEFINITE INTEGRALS:

If $g'(x)$ is continuous on $[a, b]$ and $f(x)$ is continuous on the range of $u = g(x)$, then

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

EXAMPLE:

$$\int_0^{\frac{\pi}{2}} \sin x \cos x\,dx.$$

For this example we can choose u to be either $\sin x$ or $\cos x$. It does not matter in this particular case. Let us choose:

$u = \sin x.$

Then $du = \cos x\,dx \rightarrow \quad dx = \frac{du}{\cos x}$

Let us change the limits of integration: When $x = 0,\ u = 0$. When $x = \frac{\pi}{2}, u = 1.$

This gives us: $\int_0^{\frac{\pi}{2}} \sin x \cos x\,dx = \int_0^1 \mathrm{u} \cos x\ \frac{du}{\cos x} = \int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}.$

EXAMPLE:

$\int_0^3 \frac{2x-2}{x^2-2x+1}\,dx.$

We choose $u = x^2 - 2x + 1$. Hopefully you see the numerator is the derivative of the denominator.

When $x = 0, u =$1, and when $x = 3, u = 4$.

Then $du = (2x - 2)\,dx$ gives:

$$\int_0^3 \frac{2x-2}{x^2-2x+1}\,dx = \int_3^4 \frac{1}{u}\,du = \ln|u|\Big]_3^4 = \ln 4 - \ln 3 = \ln\left(\frac{4}{3}\right)$$

EXAMPLE:

$\int_1^2 \frac{\ln x}{x}\,dx$. We choose $u = \ln x$, because $\frac{d}{dx}\ln x = \frac{1}{x}$.

When $x = 1, u = 0,$ and when $x = 2,\ u = \ln 2$

$$du = \frac{1}{x}dx \rightarrow \quad dx = x\,du$$

$$\int_1^2 \frac{\ln x}{x}\, dx = \int_0^{\ln 2} u\, du = \frac{u^2}{2}\bigg|_0^{\ln 2} = \frac{(\ln 2)^2}{2}$$

SYMMETRY IN INTEGRALS:

INTEGRALS OF EVEN AND ODD FUNCTIONS:

Let f be continuous on $[-a, a]$, then:

1) If f is even, i.e. $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$.

2) If f is odd, i.e. $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)\, dx = 0$.

Observe how handy these can be. When you recognize them, it can make calculations much easier. For number 2), the answer is immediate. For number 1), it is still shorter: Whenever a limit of integration is 0, it makes the calculation simpler (most of the time).

Also note, that these are intuitive. Even functions are symmetric about the x-axis, and odd functions are symmetric about the origin. It follows that the area of even functions will be double, and that the area of odd functions will be zero. Now we complete the formal proof:

PROOF:

We write: $\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx = -\int_0^{-a} f(x)\, dx + \int_0^a f(x)\, dx$ using property 1) for Definite Integrals in Section 2.

Next, we substitute $u = -x$, which implies $du = -dx$. (This is the method we use to find out if f is even, odd, or neither).

When $x = -a,\ u = a$.

Then, $-\int_0^{-a} f(x)\, dx = -\int_0^a f(-u)(-du) = \int_0^a f(-u)\, du$,

So, $\int_{-a}^{a} f(x)\, dx = \int_0^a f(-u)\, du + \int_0^a f(x)\, dx$.

1) When f is even, $f(-u) = f(u) \rightarrow$

$$\int_{-a}^{a} f(x)\, dx = \int_0^a f(u)\, du + \int_0^a f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(x)\, dx = 2\int_0^a f(x)\, dx.$$

2) When f is odd, $f(-u) = -f(u) \rightarrow$

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0.$$

EXAMPLE:

Let $f(x) = x^2$, an even function:

Then $\int_{-1}^{1} x^2\,dx = 2\int_{0}^{1} x^2\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$.

EXAMPLE:

Let $f(x) = x^3$, an odd function:

Then $\int_{-4}^{4} x^3\,dx = 0$, by the Symmetry Proof, and no further calculation is necessary. (Now wasn't that much easier and a big relief?!)

EXERCISES:

Use U-substitution to find the following general indefinite integrals:

1) $\int (x^2 - 7)^8 \, 2x \, dx$

2) $\int (x^3 - 2)^5 \, 3x^2 \, dx$

3) $\int (3x^2 + 9)^3 \, x \, dx$

4) $\int (4x^3 - 1)^4 \, 5x^2 \, dx$

5) $\int (x^3 - 10)^{\frac{1}{3}} \, 2x^2 \, dx$

6) $\int \sqrt{x^2 + 4} \, 3x \, dx$

7) $\int \sqrt{x^3 + 7} \, x^2 \, dx$

8) $\int \frac{1}{\sqrt{x^2 - 2}} \, 3x \, dx$

9) $\int (x^3 - x^2 + 3)^7 \, (3x^2 - 2x) \, dx$

10) $\int (2x^3 - 4x^2 + 10)^6 \, (6x^2 - 8x) \, dx$

11) $\int (x^2 - 2x)^{10} \, (x - 1) \, dx$

12) $\int (x^2 - x)^{11} \, (4x - 2) \, dx$

13) $\int \frac{3x^2 + 4x + 1}{x^3 + 2x^2 + x - 1} \, dx$

14) $\int \frac{2x - 9 - e^x}{x^2 - 9x + 1 - e^x} \, dx$

15) $\int \frac{6x + 9}{\sqrt{x^2 + 3x}} \, dx$

16) $\int e^{7x} \, dx$

17) $\int \frac{1}{3} \, e^{-2x} \, dx$

18) $\int e^{x^2} \, 2x \, dx$

19) $\int e^{3x^2} \, 4x \, dx$

20) $\int \frac{2e^{2x}-3e^{x}+3x^2}{e^{2x}-3e^{x}+x^3}\, dx$

21) $\int (3-e^{x})(3-e^{x})\, dx$

22) $\int 2\sin x \cos x\, dx$

23) $\int \sin 5x\ \, dx$

24) $\int \sin^2 x \cos x\, dx$

25) $\int \cot x\ \, dx$

26) $\int \frac{(\ln x)^3}{x}\, dx$

27) $\int \sec^2 x \tan x\, dx$

28) $\int \sqrt{\cos x}\,\sin x\, dx$

29) $\int \sec^2 x \tan^2 x\, dx$

30) $\int \sec^2 x \tan^4 x\, dx$

31) $\int (\sin x+1)\,(\cos x+2)\, dx$

32) $\int \frac{\sec^2 x+9e^{3x}}{\tan x+3e^{3x}}\, dx$

33) $\int \frac{\arctan x}{1+x^2}\, dx$

34) $\int \frac{2\sin^{-1} x}{\sqrt{1-x^2}}\, dx$

35) $\int \frac{1}{4+x^2}\, dx$ (Hint: Factor the 4 out of the denominator, then do the u-substitution)

36) $\int \sqrt{x-2}\, 3x^2\, dx$

37) $\int \sqrt{x-1}\, x^3\, dx$

Evaluate the definite integral (Hint: Use U-substitution):

38) $\int_0^1 (x^2+1)^3\, 2x\, dx$

39) $\int_0^1 (4x^3 - 9)^4 \, 12x^2 \, dx$

40) $\int_1^3 \frac{(\ln x)^2}{x} \, dx$

41) $\int_0^3 e^{2x} \, dx$

42) $\int_0^{\frac{\pi}{4}} \sin^2 x \cos x \, dx$

43) $\int_0^1 \frac{2x-3}{x^2-3x+7} \, dx$

44) $\int_0^1 2x \, e^{x^2} dx$

45) $\int_0^{\frac{\pi}{4}} \sec x \tan x \, dx$

46) $\int_0^{\frac{\pi}{3}} \sec^2 x \, dx$

47) $\int_0^{\pi} \cos^3 x \sin x \, dx$

48) $\int_0^{\frac{\pi}{4}} \sec^2 x \tan x \, dx$

49) $\int_0^1 \frac{2 \arctan x}{1+x^2} \, dx$

50) $\int_0^1 \frac{1}{x^2+9} \, dx$

51) $\int_0^1 (e^{3x} - \sin x \cos x) \, dx$ (Hint: Use a property of integrals to rewrite as two integrals)

52) $\int_1^2 (x^3 + x^2 - x)^5 (3x^2 + 2x - 1) \, dx$

53) $\int_2^3 \frac{e^x}{e^x - e} \, dx$

54) $\int_0^{\frac{1}{2}} \frac{\arcsin x}{\sqrt{1-x^2}} \, dx$

Use symmetry to evaluate the following integrals more efficiently:

55) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$

56) $\int_{-\pi}^{\pi} 2 \sin x \, dx$

57) $\int_{-100}^{100} (x^3 + x)\, dx$

58) $\int_{-1}^{1} (2x^4 + x^2)\, dx$